String-net Models

Following Alexander Kirillov Jr. - *String-net model of Turaev-Viro invariants* [arXiv:1106.6033]

Yang Yang

Insight from Levin-Wen models

- Represent vectors in a finite dimensional Hilbert space by string-diagrams (colored by a spherical fusion category) living on a honeycomb-shaped lattice, which lives on a surface (e.g. a torus).
- Represent the Hamiltonian by manipulations of the diagrams, inspired by conventional graphical calculus of the category.
- All very intuitive and heuristic.

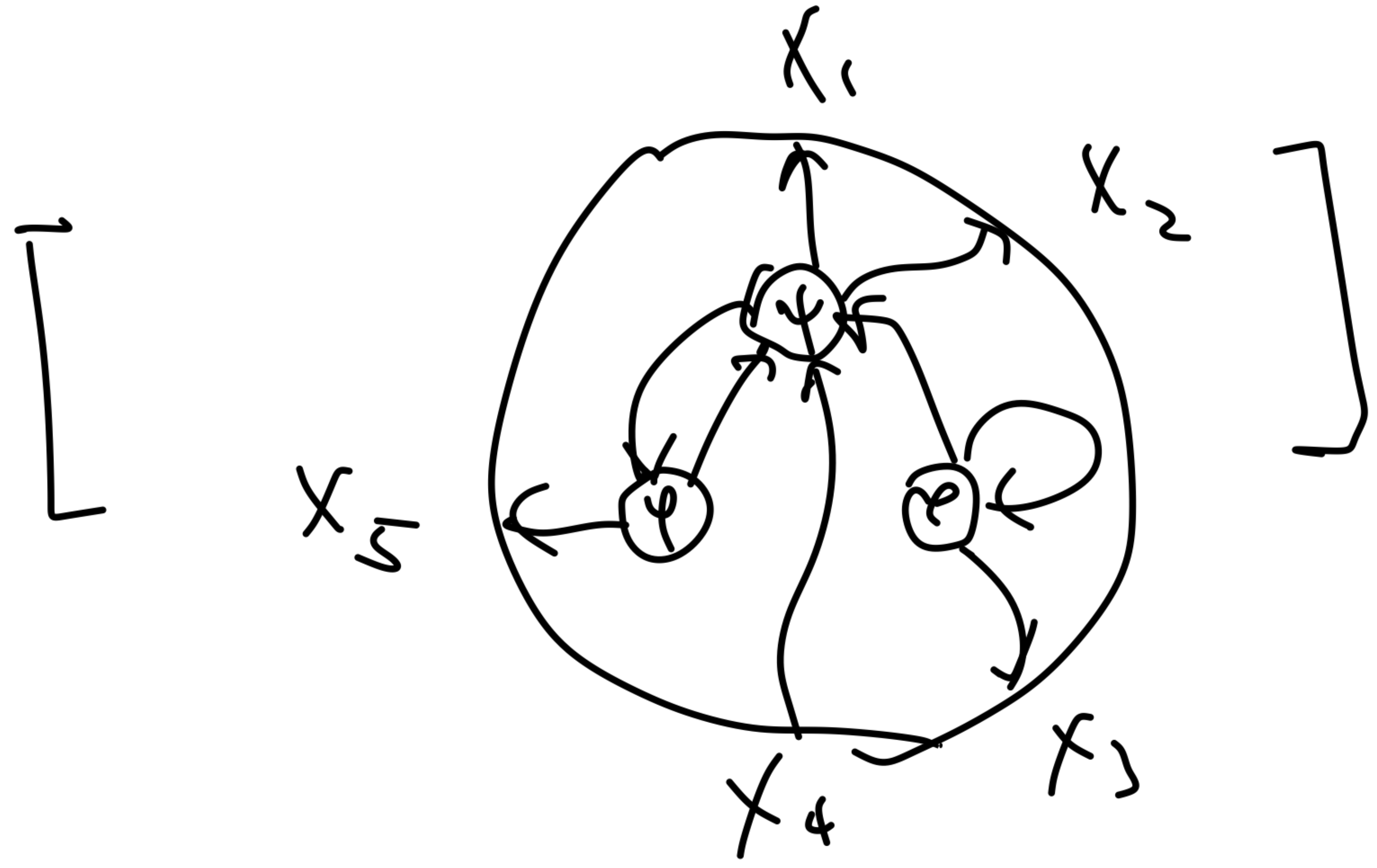
idea: graphical calculus on surfaces

String-net Models

Rigorous formulation of the idea

1. recall graphical calculus for a spherical fusion category

$\mathcal{A}$ (over $\mathbb{C}$)



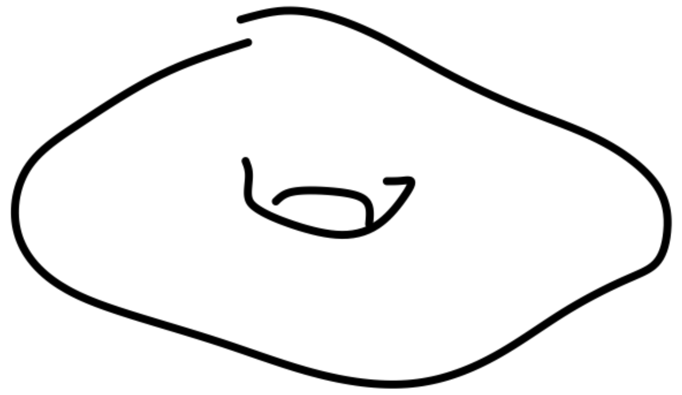
$\in \mathrm{Hom}_{\mathcal{A}}(1, X_1 X_2 X_3 X_4^{\vee} X_5)$

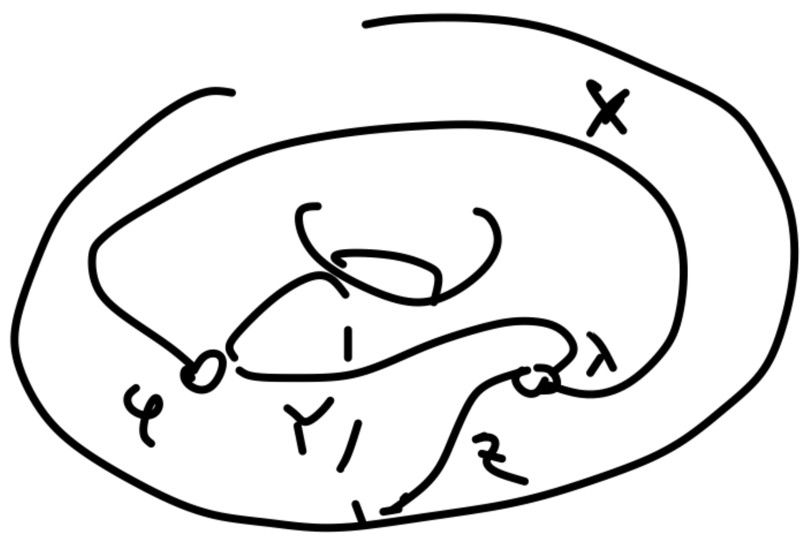
up to canonical isomorphisms

given by dualities

Given an oriented surface Σ (closed, for the moment)

e.g. a torus





Consider the set $\mathrm{Graph}_{\mathcal{A}}(\Sigma)$

$$:= \{ \text{finite } \mathcal{A}\text{-colored graphs on } \Sigma \}$$

We generate a vector space $V\mathrm{Graph}_{\mathcal{A}}(\Sigma)$

$$:= \mathrm{Span}_{\mathbb{C}}\big(\mathrm{Graph}_{\mathcal{A}}(\Sigma)\big)$$

Define the $\mathcal{A}$-string-net space for Σ

$$SN_{\mathcal{A}}(\Sigma) := VGraph_{\mathcal{A}}(\Sigma) / N_{\mathcal{A}}(\Sigma)$$

a subspace generated by

"local relations" given by $\mathcal{A}$

Precise def. of $N_{\mathcal{A}}(\Sigma)$:

A linear comb. of grps. $\Gamma = c_1 \Gamma_1 + \cdots + c_n \Gamma_n \in \mathrm{VGraph}_{\mathcal{A}}(\Sigma)$

is called a null graph if $\exists$ an embeded disc $D \hookrightarrow \Sigma$

s.t.

- Γ_i's coincide outside of D
- Γ_i's intersect ∂D transversally
- $[\Gamma \cap D]_{\mathcal{A}} = c_1 [\Gamma_1 \cap D]_{\mathcal{A}} + \cdots + c_n [\Gamma_n \cap D]_{\mathcal{A}} = 0$

evaluation according to graphical calc.

$N_{\mathcal{A}}(\Sigma) := \operatorname{Span}_{\mathbb{C}}\{\text{null graphs}\}$

then we have a surjection:

$$[\ -\ ]_{\mathcal{A}} : \mathrm{VGraph}_{\mathcal{A}}(\Sigma) \twoheadrightarrow SW_{\mathcal{A}}(\Sigma) = \mathrm{VGraph}_{\mathcal{A}}(\Sigma) \big/ N_{\mathcal{A}}(\Sigma)$$

What does $SW_{\mathcal{A}}(\Sigma)$ "look like"?

Drinfeld Center

S.F. cat. $\mathcal{A} \longmapsto Z(\mathcal{A})$ (braided, even modular)

"half-braiding"

w/ · objects : pairs $y = (Y, \gamma_y)$

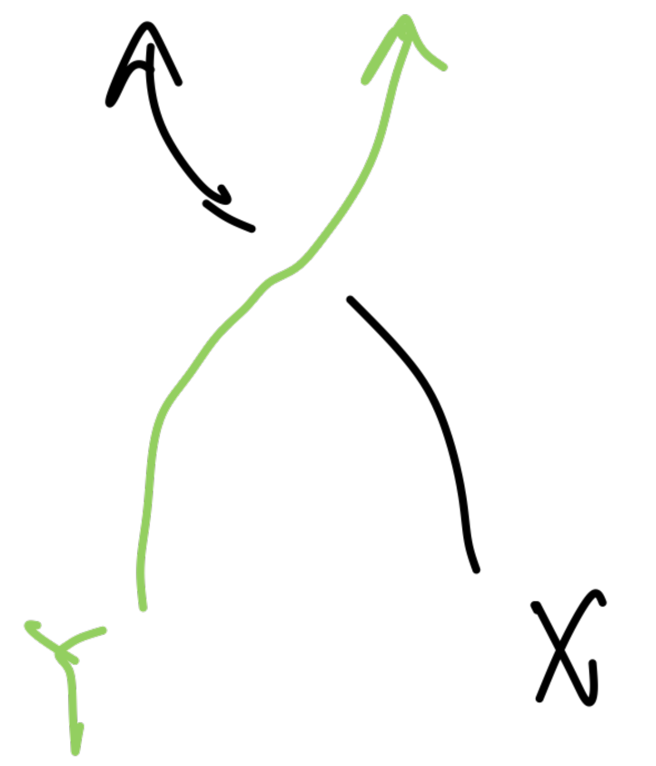
where $Y \in \mathcal{A}$, γ_y nat. iso. : $Y \otimes (-) \to (-) \otimes Y$

subjected to conditions.

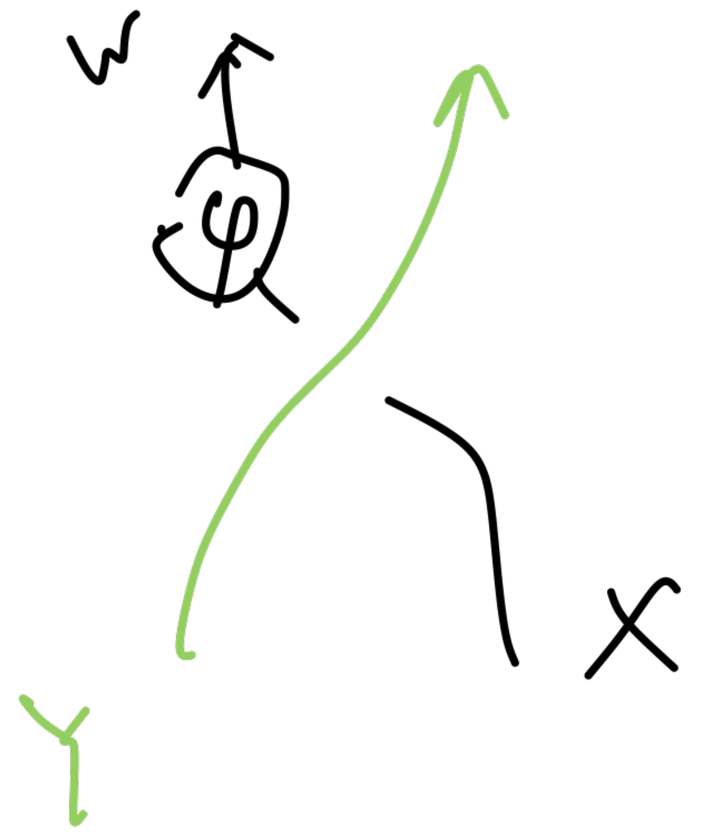
· morphisms : $f \in \mathcal{A}$ "commutes" with the half-braidings.

Diagrammatically: $Y = (y, \sigma_y : Y \otimes (-) \xrightarrow{\cong} (-) \otimes Y)$

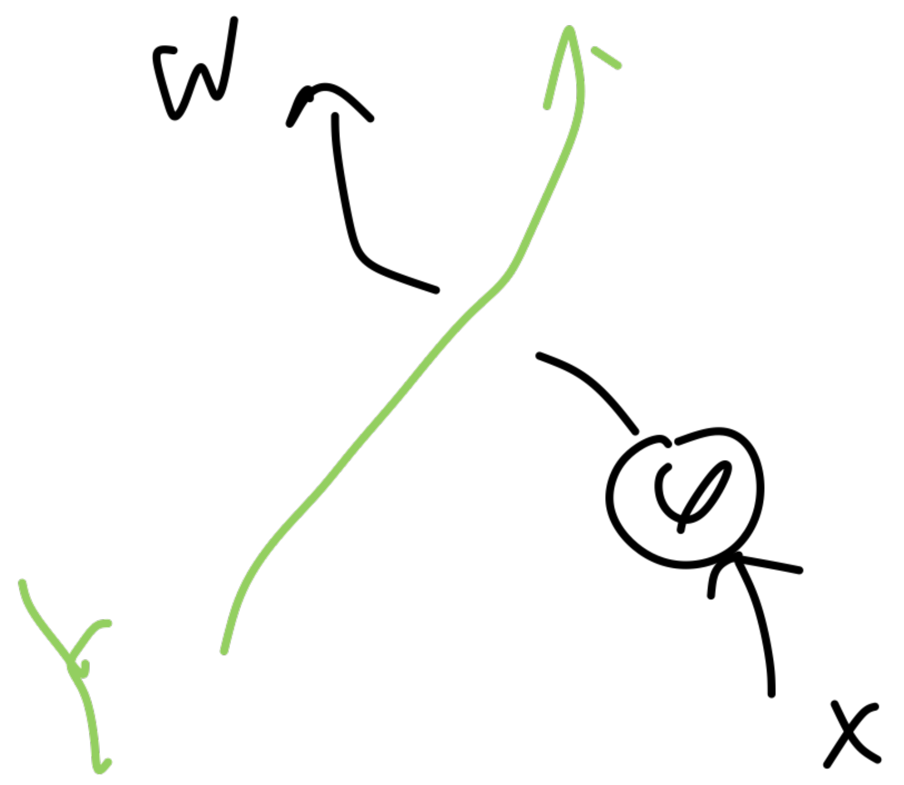
- half-braiding $(\gamma_y)_{X \in \mathcal{A}}$ =



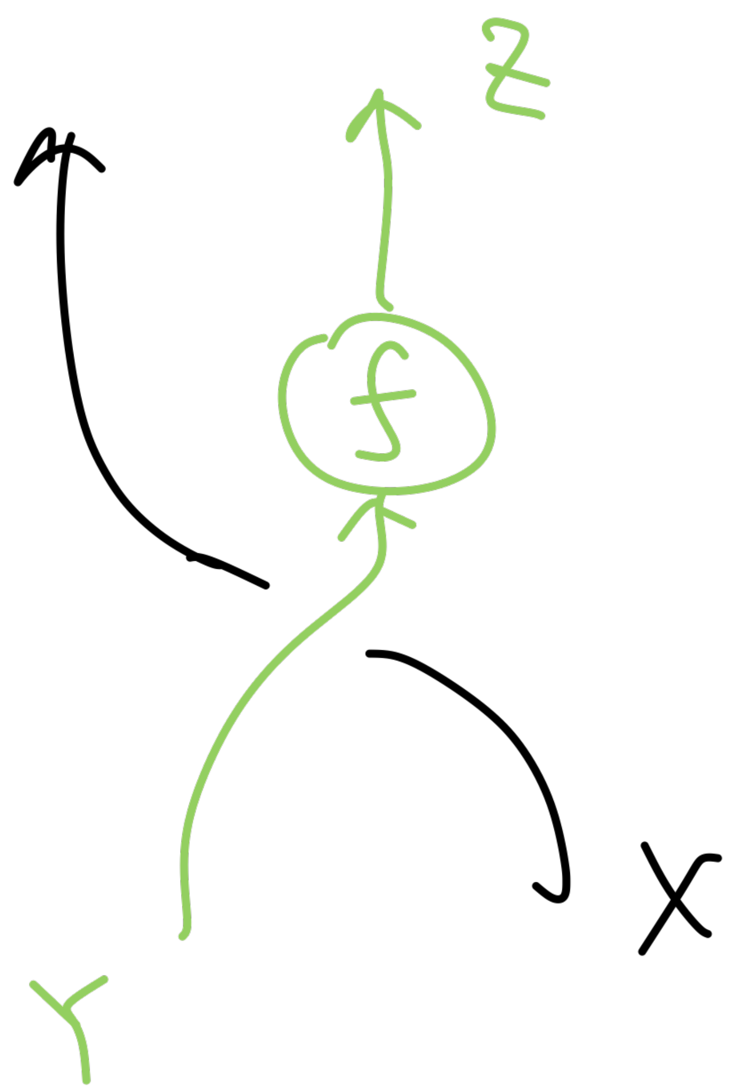
Naturality:



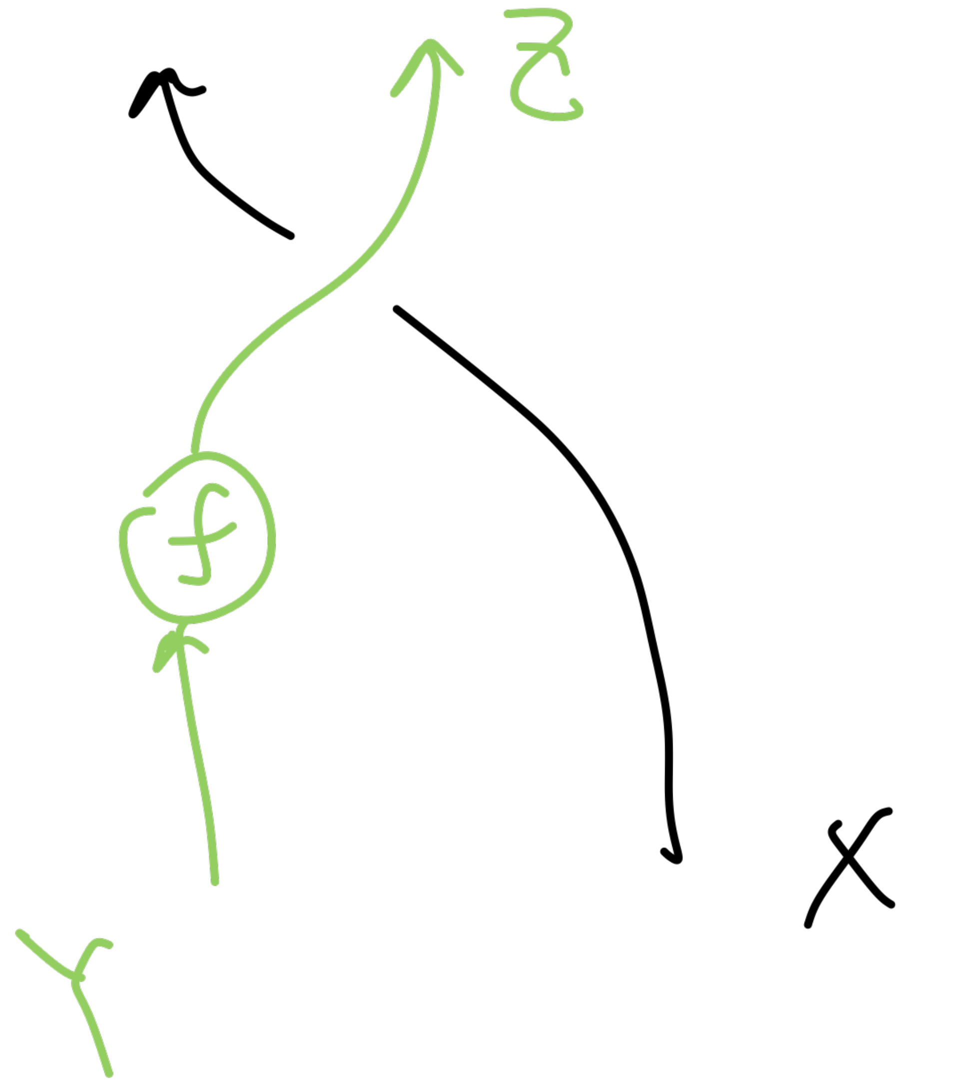
||



- condition for morphisms:



=



Examples of Drinfeld center

- $\mathcal{A} = \mathrm{Vect}_{\mathbb{Z}_2} \longrightarrow Z(\mathcal{A})$ has 4 simples
- $\mathcal{A} = H\text{-mod}$, H a Hopf alg.

$$\longrightarrow Z(\mathcal{A}) \simeq D(H)\text{-mod}$$

- $\mathcal{A}$ a MFC $\longrightarrow Z(\mathcal{A}) \underset{\text{braided}}{\simeq} \overline{\mathcal{A}} \boxtimes \mathcal{A}$

Σ^g closed surface of genus g

Theorem :

$$SN_{\mathcal{A}}(\Sigma^g) \cong \mathrm{Hom}_{Z(\mathcal{A})}(1, K^{\otimes g})$$

Where $K := \bigoplus_{\lambda \in \mathrm{simp}(Z(\mathcal{A}))/_{\mathrm{iso}}} Z_\lambda^\vee \otimes Z_\lambda$

Example: $g=1$, Σ a torus T

$$SN_{\mathcal{A}}(T) \cong \mathrm{Hom}_{Z(\mathcal{A})}\left(1, \bigoplus_{\lambda} Z_\lambda^\vee \otimes Z_\lambda\right)$$

$$\cong \bigoplus_{\lambda} \mathrm{Hom}_{Z(\mathcal{A})}(Z_\lambda, Z_\lambda)$$

$$\cong \mathbb{C}^{\oplus |\mathrm{Simp}(Z(\mathcal{A}))/\mathrm{iso}|}$$

bases: [] $_\lambda$

$$:= \sum_{\substack{i \\ \in \mathrm{Simp}(\mathcal{A})/\mathrm{iso}}} d_i \cdot \mathrm{id}_{X_i}$$ (Kirby color)