Fusion Categories

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1 Fusion Categories

1.1 Abstract Definition

Definition 1. A *fusion category* is a rigid semisimple abelian $\mathbb{C}$-linear monoidal category with only finitely many isomorphism classes of simple objects and such that the unit object is simple.

Examples.

1. Vect - the category of *finite dimensional* complex vector spaces.
2. G-Rep - the category of finite dimensional representations of a finite group G.

"monoidal category"

A *monoidal category* is a category $\mathcal{C}$ equipped with

1. a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$,
2. a unit object $1 \in \mathcal{C}$,
3. associators $(x \otimes y) \otimes z \xrightarrow{a_{x,y,z}} x \otimes (y \otimes z)$,
4. unitors $1 \otimes x \xrightarrow{\lambda_x} x$ and $x \otimes 1 \xrightarrow{\rho_x} x$,

satisfying the following *coherence relations*:

1. The pentagon identity

$$\begin{tikzcd}
& & (x \otimes y) \otimes (z \otimes w) \arrow[drr, "a_{x,y,z\otimes w}"] & & \\
((x \otimes y) \otimes z) \otimes w \arrow[urr, "a_{x\otimes y,z,w}"] \arrow[dr, "a_{x,y,z}\otimes 1_w"'] & & & & x \otimes (y \otimes (z \otimes w)) \\
& (x \otimes (y \otimes z)) \otimes w \arrow[rr, "a_{x,y\otimes z,w}"'] & & x \otimes ((y \otimes z) \otimes w) \arrow[ur, "1_x \otimes a_{y,z,w}"'] &
\end{tikzcd}$$

2. The triangle identity

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "a_{x,1,y}"] \arrow[dr, "\rho_x \otimes 1_y"'] & & (x \otimes 1) \otimes y \arrow[dl, "1_x \otimes \lambda_y"] \\
& x \otimes y &
\end{tikzcd}$$

Theorem 2 (McLane)**.** *Any two (formal) compositions of a, λ, ρ are equal.*

We use *string diagrams* to represent objects and morphisms of monoidal categories.

"$\mathbb{C}$-linear abelian"

- All Hom-sets $\mathcal{C}(x, y)$ carry the structure of a $\mathbb{C}$ vector space and composition is $\mathbb{C}$-linear.
- $\mathcal{C}$ behaves like a category of modules over a commutative ring. In particular we have direct sums, kernels and cokernels and they behave in the usual way.

"semisimple"

This means that every object of $\mathcal{C}$ can be written as a finite direct sum of simple objects. An object x is called *simple* if $\mathrm{End}(x) = \mathbb{C}1_x$.

Lemma 3 (Schur)**.** *If x, y are simple objects that are not isomorphic then $\mathcal{C}(x, y) \cong 0$.*

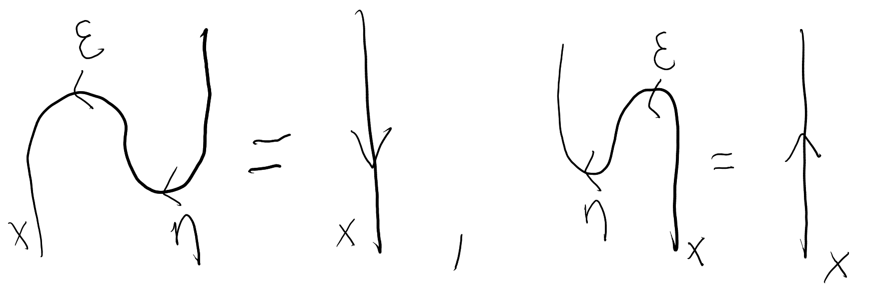
- We write I for the set of isomorphism classes of simple objects and we pick representative simple objects $e_1, \ldots, e_n \in \mathcal{C}$, where $n = |I|$.

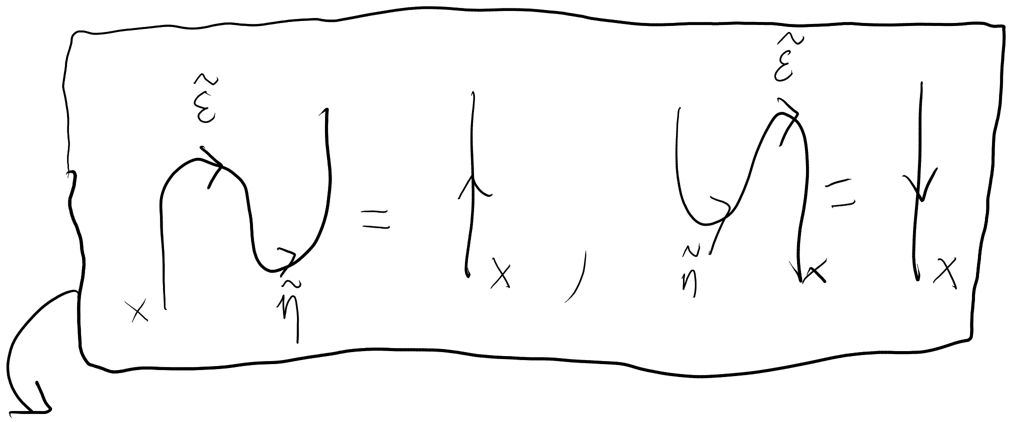
- It follows from semisimplicity that $\mathcal{C} \simeq \mathrm{Vect}^n$ as categories. However, the tensor product $\otimes$ of $\mathcal{C}$ is *not* determined by this equivalence.
- Let $x, y \in \mathcal{C}$. After choosing bases for the spaces $\mathcal{C}(e_i, x)$ and $\mathcal{C}(e_i, y)$ for all $i \in I$ we can think of morphisms $x \longrightarrow y$ as block matrices.
- In Vect the only simple object is the ground field $\mathbb{C}$.

"rigid"

This means that all objects of $\mathcal{C}$ have *left and right duals.*

- A left dual for an object $x \in \mathcal{C}$ is an object ${}^*x$ together with morphisms $\eta_x : 1 \longrightarrow x \otimes {}^*x$ and $\varepsilon_x : {}^*x \otimes x \longrightarrow 1$ such that the *snake identities* hold:

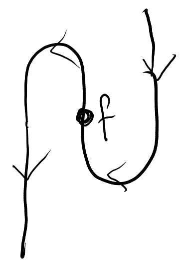




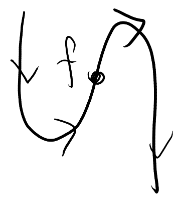
- A right dual for x is an object x^* together with morphisms $\tilde{\eta}_x : 1 \longrightarrow x^* \otimes x$ and $\tilde{\varepsilon}_x : x \otimes x^* \longrightarrow 1$ such that the snake identities hold:

- Left and right duality structures are (up to canonical isomorphisms) unique if they exist.
- Left and right duality define monoidal functors ${}^*(-), (-)^* : \mathcal{C}^{\mathrm{op,rev}} \longrightarrow \mathcal{C}$.

${}^*f =$



$f^* =$



- For all x there exists some isomorphism ${}^*x \cong x^*$.
- In Vect left and right duals agree and are given by the dual vector space. Here ε is the evaluation pairing and η picks out the tensor corresponding to the identity map under $V^* \otimes V \cong \mathrm{End}(V)$.
- Rigidity can be seen as a finiteness condition. A vector space has a dual in this sense if and only if it is finite dimensional.

Example.

1. The category Vect_G^ω of G-graded vector spaces, for G a finite group and $\omega \in H^3(G, \mathcal{C}^\times)$. It has simple objects $\mathbb{C}_g$, $g \in G$ and $\mathbb{C}_g \otimes \mathbb{C}_h \cong \mathbb{C}_{gh}$. The associators (up to equivalence) are determined by ω.
2. Many examples come from representations of Hopf Algebras and of vertex operator algebras.

1.2 Fusion Categories in Coordinates

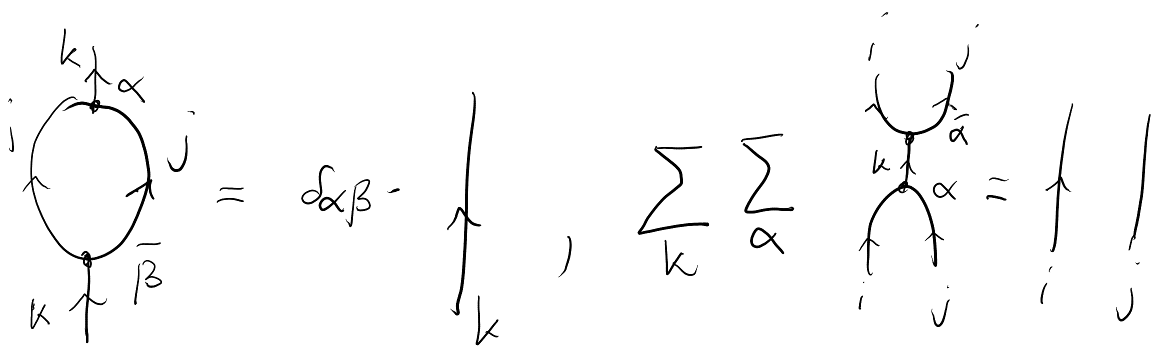
Fusion Coefficients

- The *fusion coefficients* or *fusion rules* of $\mathcal{C}$ are the non-negative integer numbers $N_{ij}^k := \dim_{\mathbb{C}} \mathcal{C}(e_k, e_i \otimes e_j)$, for $i, j, k \in I$.
- By taking isomorphism classes we obtain the *Groethendieck ring* $\mathbb{C}[\mathcal{C}]$ of $\mathcal{C}$. Addition corresponds to $\oplus$ and multiplication to $\otimes$. It has a basis consisting of the classes of simple objects and it is completely determined by the N_{ij}^k.
- The associativity of the multiplication amounts to

$$\sum_\nu N_{ij}^\nu N_{\nu k}^l = \sum_\nu N_{i\nu}^l N_{jk}^\nu =: N_{ijk}^l.$$

F-matrices

We fix once and for all bases $\{\lambda_{ijk}^\alpha\}$ for $\mathcal{C}(e_i \otimes e_j, e_k)$ and dual bases $\{\lambda_{\bar{\alpha}}^{ijk}\}$ for $\mathcal{C}(e_k, e_i \otimes e_j)$.



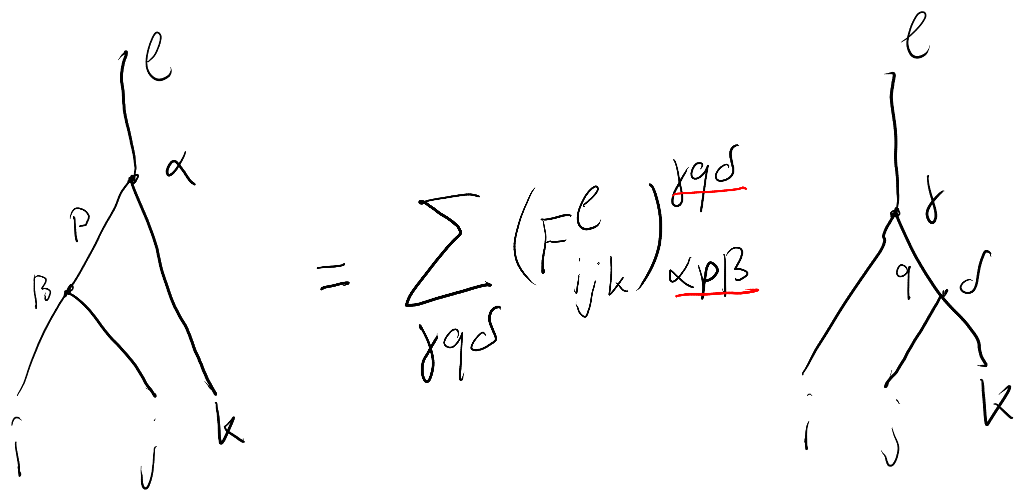
- For fixed indices i, j, k, l the decompositions

$$\mathcal{C}(e_i \otimes (e_j \otimes e_k), e_l) \cong \bigoplus_\nu C(e_j \otimes e_k, e_\nu) \otimes \mathcal{C}(e_i \otimes e_\nu, e_l)$$

and

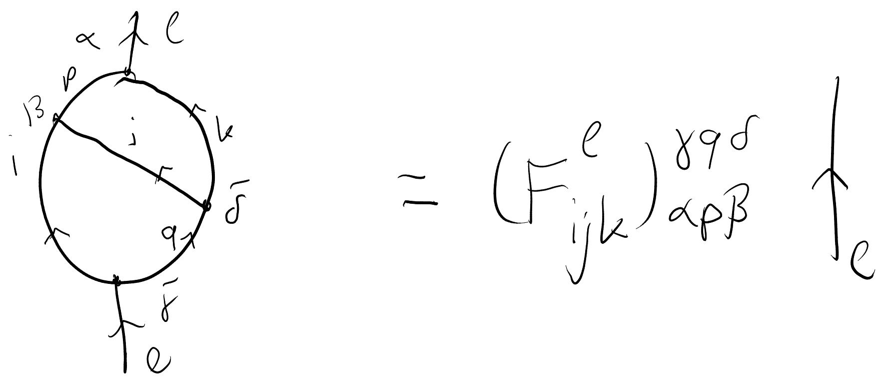
$$\mathcal{C}((e_i \otimes e_j) \otimes e_k, e_l) \cong \bigoplus_{\nu} \mathcal{C}(e_i \otimes e_j, e_\nu) \otimes \mathcal{C}(e_\nu \otimes e_k, e_l)$$

give rise to two bases of $\mathcal{C}((e_i \otimes e_j) \otimes e_k, e_l)$:



The coefficients of the base change are called the *F-symbols* and are the entries of the *F-matrices*.

- They encode the data of the associators and can be computed as



- The pentagon identity in coordinates becomes the system of equations

$$\sum_{\delta}(F^m_{ijq})^{\delta r\varepsilon}_{\alpha p\beta}(F^m_{rkl})^{\nu s\mu}_{\delta q\gamma} = \sum_{\xi,t,\eta,\sigma}(F^p_{jkl})^{\xi t\eta}_{\beta q\gamma}(F^m_{itl})^{\nu s\sigma}_{\alpha p\xi}(F^t_{ijk})^{\mu r\varepsilon}_{\sigma t\eta}$$

for all $\nu, s, \mu, r, \varepsilon$ and $\alpha, p, \beta, q, \gamma$.

Lemma 4. *A fusion category is uniquely determined by its fusion rules N^k_{ij} and F-symbols $(F^l_{ijk})^{\gamma q\delta}_{\alpha p\beta}$.*

Examples.

1. The Fibonacci category has two simple objects, 1 and τ, satisfying $\tau \otimes \tau \cong 1 \oplus \tau$. The F-symbols are given by

See e.g. • [Bruce Bartlett - Fusion Categories via String Diagrams, p.6]

• [Rowell, Stong, Wang - On Classification of Modular Tensor Categories, p.36-p.37]

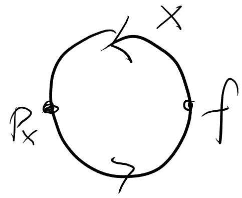
2. $\mathbb{Z}_2 - \mathrm{Vect}$ has two simple objects e_0 and e_1 and identity associators.

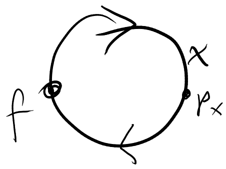
1.3 Traces and Dimensions

Definition 5. A pivotal structure on $\mathcal{C}$ is an isomorphism of monoidal functors ${}^*(-) \cong (-)^*$. This means that we have isomorphisms $p_x : {}^*x \cong x^*$ for all x that satisfy extra coherence conditions.

- Conjecture: Every Fusion category allows a pivotal structure. This is true for all known examples.

- Given a pivotal fusion category $\mathcal{C}$ the left- and right *traces* of $f \in \mathrm{End}(x)$ are defined by

$\mathrm{tr}_l(f) =$ 

$\mathrm{tr}_r(f) =$ 

We will henceforth assume $\mathrm{tr}_l(f) = \mathrm{tr}_r(f) =: \mathrm{tr}(f)$ for all f. This property of $\mathcal{C}$ is known as *sphericality*.

- For $x \in \mathcal{C}$ we define its *quantum dimension* as $\dim(x) = \mathrm{tr}(1_x)$. We have $d_i := \dim(e_i) \neq 0$ for all simple objects e_i. We have $\mathrm{tr}(fg) = \mathrm{tr}(gf)$ $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\mathrm{tr}(g)$. The quantum dimension gives an algebra homomorphisms $\mathbb{C}[\mathcal{C}] \longrightarrow \mathbb{C}$. It follows from sphericality that we have $\dim(x^*) = \overline{\dim(x)}$.

- There exists a unique algebra homomorphism $\mathbb{C}[\mathcal{C}] \longrightarrow \mathbb{C}$ that takes only positive real values. It assigns a numbers d_i^+ to each simple object e_i, its so called *Frobenius-Perron dimension*, satisfying
$$d_i^+ d_j^+ = \sum_k N_{ij}^k d_k^+.$$

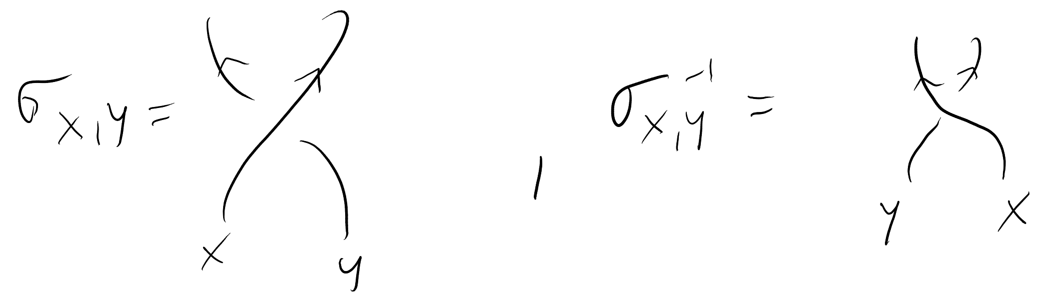
- The global dimension of $\mathcal{C}$ is given by $\mathcal{D}(\mathcal{C}) := \sum_{i \in I} |d_i|^2 \in \mathbb{R}_{>0}$ and the Frobenius-Perron dimension $\mathcal{D}^+(\mathcal{C}) := \sum_{i \in I} (d_i^+)^2$. They satisfy $\mathcal{D}(\mathcal{C}) \leq \mathcal{D}^+(\mathcal{C})$. If this is an equality we say that $\mathcal{C}$ is *pseudo-unitary*.

Modular Fusion Categories

Definition 6. A *modular fusion category* (called modular tensor categories in the non-Hamburg literature) is a ribbon fusion category with an invertible s-matrix.

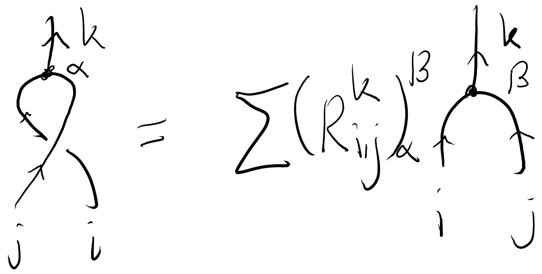
“ribbon”

- This means that we have a *braiding* and a *twist*.
- A braiding amounts to a natural isomorphism $\sigma_{x,y} : x \otimes y \xrightarrow{\cong} y \otimes x$ denoted graphically as

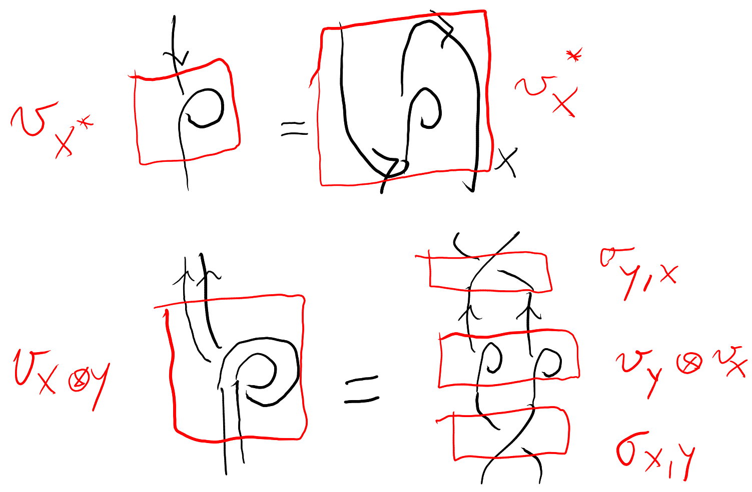


The $\sigma_{x,y}$ are required to satisfy coherence equations that read graphically as follows:

- The R-matrices are defined component wise by



- A *t*wist is a natural family of isomorphisms $\theta_x : x \xrightarrow{\cong} x$, satisfying



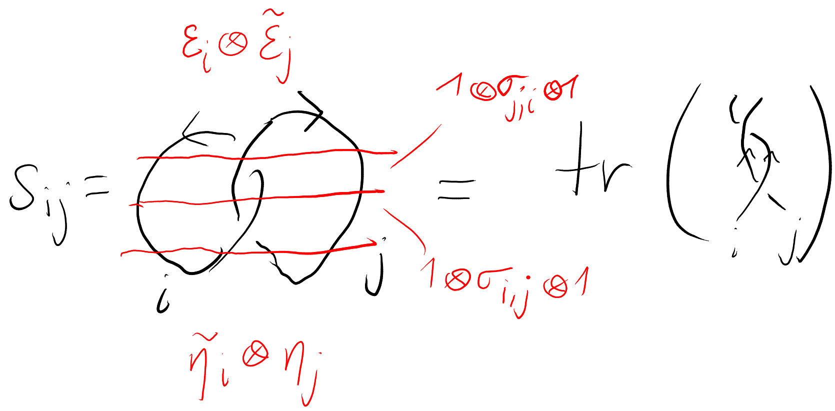
- Using these relations one can show that

$$\sum_{\beta}(R^k_{ij})^{\beta}_{\alpha}(R^k_{ji})^{\gamma}_{\beta} = \frac{\theta_k}{\theta_i\theta_j}\delta_{\alpha,\gamma}$$

- A ribbon category comes endowed with a canonical pivotal structure, defined using braiding and twist.

s-matrices and Modularity

- The s-matrices are defined by



- They can be computed by

$$s_{ij} = \sum_k \frac{\theta_k}{\theta_i\theta_j} N^k_{ij} d_k.$$

- $\mathcal{C}$ is called *modular* if its s-matrix is invertible.

- The "modular" in "modular fusion category" comes from a (projective) action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ that is generated by the matrices s and $t = \mathrm{diag}(\theta_i)$.

Examples. 1. $\mathbb{Z}_2$-Vect has trivial R-matrices and can not be endowed with a modular structure. However with a different braiding and non trivial associator this is possible. (Semion MFC.)

2. Fibonacci-category has twist $\theta_1 = 1$, $\theta_\tau = e^{\frac{4\pi i}{5}}$, braiding $R_1^{\tau\tau} = e^{-\frac{4\pi i}{5}}$, $R_1^{\tau\tau} = e^{\frac{3\pi i}{5}}$.

Drinfeld Center

Definition 7.

- The Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a fusion category $\mathcal{C}$ is a higher categorical analogue of the center of an algebra.
- The objects of $\mathcal{Z}(\mathcal{C})$ are pairs (x, β) where x is an object of $\mathcal{C}$ and β is a natural family of isomorphism $\beta_y : x \otimes y \xrightarrow{\cong} y \otimes x$ such that $\beta_{y \otimes z} = (1_y \otimes \beta_z) \circ (\beta_y \otimes 1_z)$
- A morphism $(x, \beta) \longrightarrow (y, \varphi)$ is a morphism $f \in \mathcal{C}(x, y)$ such that for all $z \in \mathcal{C}$ we have $(1_z \otimes f) \circ \beta_z = \varphi_z \circ (f \otimes 1_z)$.
- The tensor product is given by $(x, \beta) \otimes (y, \varphi) = (x \otimes y, (\beta \otimes 1_y) \circ (1_x \otimes \varphi))$
- The Drinfeld-Center of a fusion category is modular.
- If $\mathcal{C}$ is already a modular fusion category then $\mathcal{Z}(\mathcal{C}) \equiv \mathcal{C} \boxtimes \tilde{\mathcal{C}}$.
- $\mathcal{Z}(\mathbb{Z}_2 - \mathrm{Vect}) = (\mathbb{Z}_2 \times \mathbb{Z}_2) - \mathrm{Vect}$.

References:

- Bruce Bartlett – Fusion Categories via String Diagrams arxiv: 1502.02882
- Rowell, Stong, Wang – On Classification of Modular Tensor Categories arxiv: 0712.1377
- Etingof, Gelaki, Nikshych, Ostrik – Tensor Categories
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