ZMP Seminar: Harmonic metric and NAH

1. Three worlds and three cohomology theories
2. "Nonabelianising" the three cohomology theories
3. Harmonic metric, Chern Laplacian, Hitchin's eqns
4. Nonabelian Hodge correspondence
5. Example

§1. Three worlds

Three worlds: topological spaces $\supset$ smooth manifolds $\supset$ complex manifolds

Three cohomology theories: "Betti" "de Rham" "Dolbeault"

We'll introduce these three cohomology theories, comment on their relationships to one another and reformulate to make clear what is abelian about them. Let X be a compact top. space.

Betti: $H^k_B(X;\mathbb{C})$, just the singular cohomology

de Rham: $H^k_{dR}(X;\mathbb{C})$, diff. forms, exterior derivative d
smooth mfld

$\boxed{\text{Thm (de Rham)}}$: For X a compact smooth manifold
$H^k_B(X;\mathbb{C}) \cong H^k_{dR}(X;\mathbb{C})$.

complex mfld
Dolbeault: $H^q(X;\Omega^p)$, diff. forms taking values in holomorphic forms, Dolbeault operator $\bar{\partial}$.

~~Dolbeault-Hodge theorem~~

$\boxed{\text{Thm (Dolbeault-Hodge)}}$ For X a compact Kähler manifold, $H^k_{dR}(X;\mathbb{C}) \cong \bigoplus_{k=p+q} H^q(X;\Omega^p)$

For the rest of this talk, X will be a (cpt.) complex curve i.e. a Riemann surface, so it is automatically Kähler.

$$\text{Betti} \xleftrightarrow{\text{de Rham}} \text{de Rham} \xleftrightarrow{\text{Dolbeault Hodge}} \text{Dolbeault}$$

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What is abelian about this?
Specialise to $k=1$ i.e. we ~~we~~ always work with the first cohomology group in whatever cohomology theory we are interested in.

Betti: $\mathrm{Hom}(\pi_1 X, GL(1,\mathbb{C}))$ i.e. $GL(1,\mathbb{C})$-rep'ns of $\pi_1 X$
abelian

Why? Because $H^1_B(X)$ is an abelianisation of $\pi_1 X$.

For de Rham and Dolbeault, we introduce a ~~complex~~ trivial complex line bundle $E \to X$, which is to say a $GL(1,\mathbb{C})$-bundle $E \to X$ with $c_1(E) = 0$.

de Rham: Flat ~~connections~~ bundles (E, ∇) modulo some notion of "gauge transformations".
connection

Why? Connections ∇ on a trivial line bundle may be identified with 1-forms A. (so $\nabla = d + A$).

$\nabla^2 = 0 \Leftrightarrow dA = 0$. So A is closed

Interpret gauge transformations as $A \mapsto A + df$
So flat bdls mod. g.t. are closed 1-forms / exact 1-forms.

$\boxed{\text{Def.}}$ A (set of) $\mathbb{C}$-linear map(s) $\bar\partial_E : \Omega^{\bullet,\bullet}(E) \to \Omega^{\bullet,\bullet+1}(E)$
s.t. $\bar\partial_E(\underset{E}{v} \otimes \underset{\Omega^{\bullet,\bullet}}{\alpha}) = v \otimes \bar\partial\alpha + (\bar\partial_E v)\wedge\alpha$ is said to be a $\bar\partial$-connection.

$\boxed{\text{Def.}}$ A (set of) $\mathbb{C}$-linear map(s) $\partial_E : \Omega^{\bullet,\bullet}(E) \to \Omega^{\bullet+1,\bullet}(E)$
s.t. $\partial_E(v \otimes \alpha) = v \otimes \partial\alpha + (\partial_E v)\wedge\alpha$ is said to be a ∂-connection.

$\boxed{\text{Rem.}}$ $\bar\partial$-connection is the $(0,1)$ part of a usual connection and ∂-connection is the $(1,0)$ part of a usual connection. In particular ~~putting~~ ~~a $\bar\partial$-connecti~~ $\partial_E + \bar\partial_E$ is a connection.

$\boxed{\text{Def.}}$ A holomorphic structure $\bar\partial_E$ on $E \to X$ is a $\bar\partial$-connection s.t. $\bar\partial_E^2 = 0$.

Intuitive idea: we can find holomorphic transition functions.

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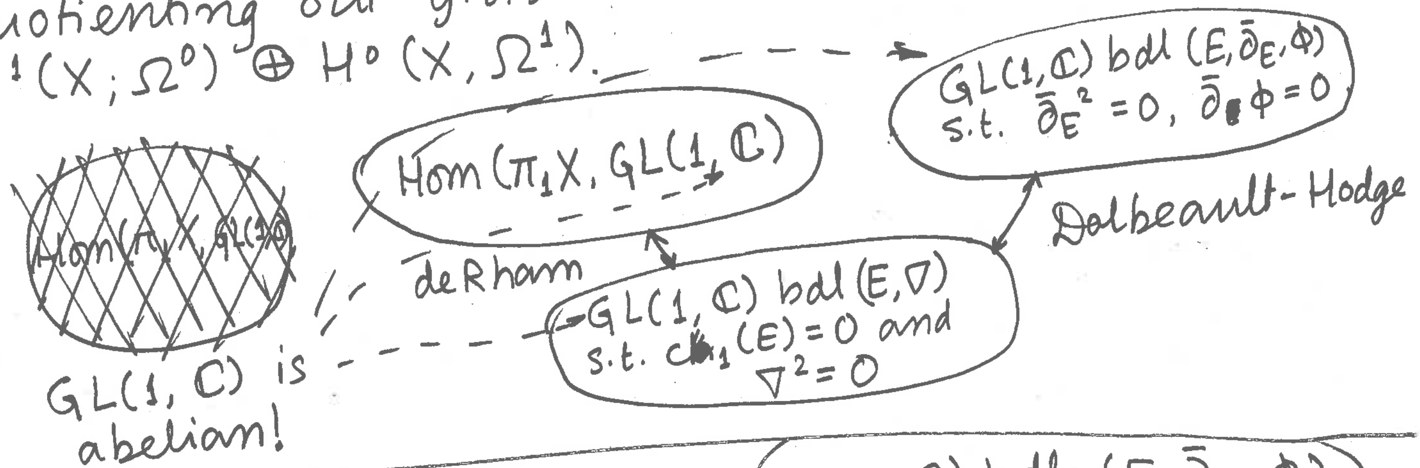
<u>Dolbeault</u>: Bundles $(E, \bar\partial_E, \phi)$ with holomorphic structure $\bar\partial_E$ and $(1,0)$-form ϕ s.t. $\bar\partial\phi = 0$ modulo some notion of gauge transformations.

Why? Holomorphic structures $\bar\partial_E$ on a trivial line bundle may be identified with $(0,1)$-forms $\bar A$ $(\text{so } \bar\partial_E = \bar\partial + \bar A)$. Then, $\bar\partial_E^2 = 0 \Leftrightarrow \bar\partial \bar A = 0$.

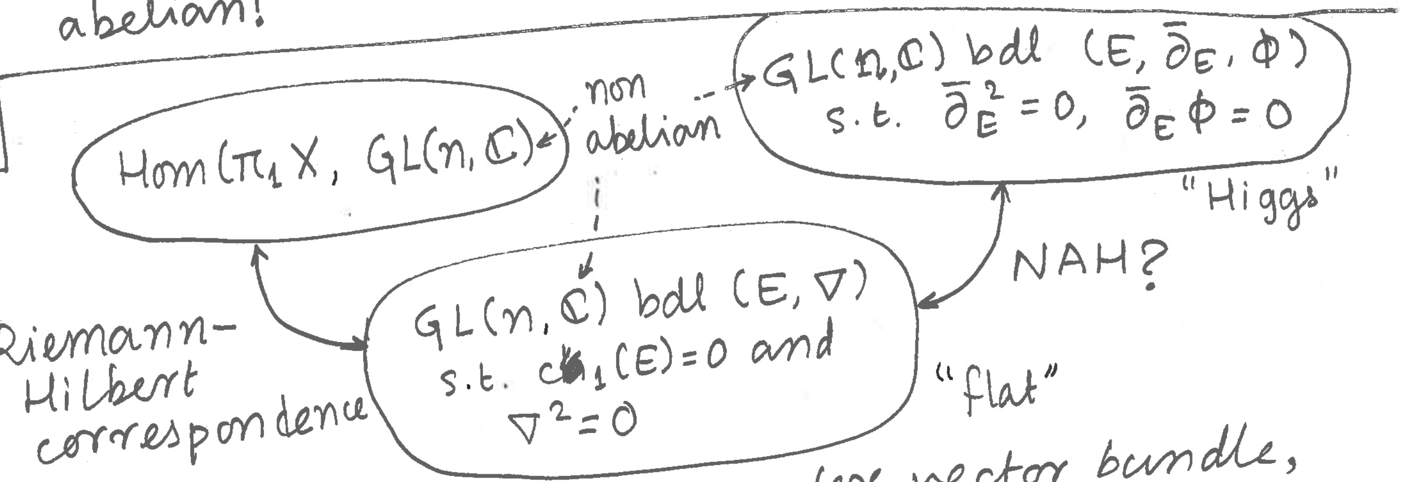
So, $(\bar A, \phi) = (\bar\partial\text{-closed } (0,1)\text{-form}, \bar\partial\text{-closed } (1,0)\text{-form})$

Interpret gauge transformations to be $(\bar A, \phi) \mapsto (\bar A + \bar\partial f, \phi)$.

Quotienting out g.t.s $\Rightarrow$ element in ~~$H^1(X, \Omega^0)$~~ $H^1(X; \Omega^0) \oplus H^0(X, \Omega^1)$.



§2.



Now E is a rank n complex vector bundle, and ϕ is an $(\text{End } E)$-valued $(1,0)$-form.

$\boxed{\text{Rem}}$ when E is a line bundle $\text{End } E = \mathbb{C}$ and $\bar\partial_E \phi = \bar\partial\phi$.

$\boxed{\text{Def}}$ A <u>Higgs bundle</u> $(E, \bar\partial_E, \phi)$ is a (holomorphic) bundle $E \to X$ with holomorphic structure $\bar\partial_E$ and $\phi \in \Omega^{1,0}(\text{End } E)$ such that $\bar\partial_E \phi = 0$.

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Riemann-Hilbert correspondence

$GL(n,\mathbb{C})$ rep'ns of $\pi_1 X \Leftrightarrow$ flat bundles (E,∇)

$GL(n,\mathbb{C})$ rep'n $\Rightarrow$ flat bdl :

Let ρ be the rep'n given. Construct the following bundle $(\tilde{X} \times \mathbb{C}^n)/\pi_1 X$ where $\pi_1 X$ acts on universal cover $\tilde{X}$ via deck transformations and on $\mathbb{C}^n$ via ρ. The ~~#~~ connection upstairs is the usual derivative on $\mathbb{C}^n$ (it's trivial). Since it is $\pi_1 X$-invariant it descends downstairs. This defines a flat connection downstairs.

Flat bdl $\Rightarrow$ $GL(n,\mathbb{C})$ rep'n :

~~Look at~~ The holonomies around ~~no~~ loops furnish a $GL(n,\mathbb{C})$ rep'n of $\pi_1 X$.

The two maps are inverses of each other. So it's a one-one correspondence.

Nonabelian Hodge correspondence

$$\text{Flat bdls} \xleftrightarrow[\text{not quite true}]{?} \text{Higgs bdls} \quad (ch_1 E = 0)$$

what is true is

$$\text{Subset of flat bdls} \longleftrightarrow \text{subset of Higgs bdls} \quad (ch_1 E = 0)$$

More precise version of this in next section.

§3.

$$\text{Flat bdls + "harmonic metric"} \underset{\text{Hitchin's eqns}}{\overset{\text{chern Laplacian}}{\rightleftarrows}} \text{Higgs bdls } (ch_1 E = 0) \text{ + "harmonic metric"}$$

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$\boxed{\text{Lemma}}$ Given a hermitian metric K on E and a $\bar\partial$-connection $\bar D$, there exists a unique connection D_K (called the Chern connection) s.t. D_K is unitary w.r.t. K (i.e. $d(K(v,w)) = K(D_K v, w) + K(v, D_K w)$) and $D_K^{0,1} = \bar D$.

$\boxed{\text{Cor.}}$ Given a hermitian metric K on E and a ∂-connection D, there exists a unique connection D_K s.t. D_K is unitary w.r.t. K and $D_K^{1,0} = D$.

$\boxed{\text{Rem.}}$ The above means that for every $\bar\partial$-connection $\bar D$ we can find a unique ∂-connection D s.t. $D + \bar D$ is unitary. Conversely, for every ∂-connection D we can find a unique $\bar\partial$-connection s.t. $D + \bar D$ is unitary.

Flat bundles to (possibly) Higgs bundles

Data: connection ∇ s.t. $\nabla^2 = 0$, hermitian metric K.

Separate out $(1,0)$ and $(0,1)$ parts: $\nabla = \underset{\partial\text{-conn.}}{\nabla^{1,0}} + \underset{\bar\partial\text{-conn.}}{\nabla^{0,1}}$

$\exists!$ $\bar\partial$-connection $\delta^{0,1}$ s.t. $\nabla^{1,0} + \delta^{0,1}$ is unitary wrt K.

$\exists!$ ∂-connection $\delta^{1,0}$ s.t. $\nabla^{0,1} + \delta^{1,0}$ is unitary wrt K.

~~Define~~ Set $\bar\partial_E = \frac{1}{2}(\nabla^{0,1} + \delta^{0,1})$, $\partial_E = \frac{1}{2}(\nabla^{1,0} + \delta^{1,0})$

$\phi = \frac{1}{2}(\nabla^{1,0} - \delta^{1,0})$, $\bar\Phi = \frac{1}{2}(\nabla^{0,1} - \delta^{0,1})$

Desiderata: $\bar\partial_E^2 = 0$, $\bar\partial_E \phi = 0$.

Define $d_E := \partial_E + \bar\partial_E$, $\Psi := \phi + \bar\Phi$,

$\Delta_E := d_E d_E^* + d_E^* d_E$ — formal adjoint w.r.t. (some choice of) metric on X.

$$\boxed{\nabla^2 = 0,\ \bar\partial_E^2 = 0,\ \bar\partial_E\phi = 0} \Leftrightarrow \boxed{\nabla^2 = 0,\ \Delta_E \Psi = 0}$$

$\boxed{\text{Def.}}$ Metric K (on E) for which this holds is said to be harmonic metric for flat bdl (E, ∇).

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Higgs bundles (with vanishing first Chern class) to (possibly) flat bundles

Data: holomorphic structure $\bar\partial_E$, $\phi \in \Omega^{1,0}(\mathrm{End}\,E)$ s.t. $\bar\partial_E \phi = 0$, hermitian metric K.

$\exists!$ ~~∂_E~~ a ∂-connection ∂_E s.t. $d_E := \partial_E + \bar\partial_E$ is unitary w.r.t. K.

$\exists!$ $\bar\Phi \in \Omega^{0,1}(\mathrm{End}\,E)$ s.t. $K(\phi v, w) = K(v, \bar\Phi w)$
(in local coordinates, if $\phi = M_i\, dz^i$ where M_i is a matrix, then $\bar\Phi = M_i^{\dagger}\, d\bar z^i$ where $\dagger$ denotes hermitian conjugate wrt K)

Set $\nabla = d_E + \phi + \bar\Phi$

Desideratum: $\nabla^2 = 0$

Define $F_E := d_E^2$ (curvature of Chern connection)

$$\boxed{\begin{gathered}\bar\partial_E \phi = 0\\ \nabla^2 = 0\end{gathered}} \iff \boxed{\begin{gathered}F_E + [\phi, \bar\Phi] = 0\\ \bar\partial_E \phi = 0\end{gathered}} \quad \text{Hitchin's equation!}$$

(also $\bar\partial_E^2 = 0$) (also $\bar\partial_E^2 = 0$)

~~Rem. The above two procedures are inverses~~

Def. Metric K (on E) for which this holds is said to be harmonic metric for Higgs bundle $(E, \bar\partial_E, \phi)$. ~~with $ch_1(E) = 0$~~

Rem. The two procedures above are inverses of each other.

Rem. We need $c_1(E) = 0$ because $c_1(E)$ is the obstruction to existence of flat connections on E.

§4.

Flat bundles that admit harmonic metrics $\longleftrightarrow$ Higgs bundles with $c_1(E) = 0$ that admit harmonic metrics

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Q. When do flat bundles admit harmonic metrics?

Q. When do Higgs bundles with vanishing first Chern class admit harmonic metrics?

$\boxed{\text{Def.}}$ A bdl ~~E/X~~ with conn. (E, ∇) that is the direct sum of irreducible bundles with connection is said to be <u>semisimple</u>.

$\boxed{\text{Def.}}$ <u>The slope of a bundle</u> E (over a curve X), $\mu(E)$ is defined to be $\mu(E) = \frac{c_1(E)}{rk(E)}$.

$\boxed{\text{Def.}}$ A Higgs bundle $(E, \bar{\partial}_E, \phi)$ is said to be <u>stable</u> iff for every ~~ϕ-invari~~ proper ϕ-invariant subbundle $F \subset E$, $\mu(F) < \mu(E)$. (so, in particular, for us this means $\mu(F) < 0$, i.e. $ch_1(F) < 0$.)

$\boxed{\text{Def.}}$ A Higgs bundle $(E, \bar{\partial}_E, \phi)$ is said to be <u>polystable</u> iff it is the direct sum of stable Higgs bundles.

$\boxed{\text{Thm (Corlette, Donaldson)}}$ A flat bundle (on curve X) admits a harmonic metric iff it is semisimple.

$\boxed{\text{Thm (Hitchin, Simpson)}}$ A Higgs bundle (on curve X) with vanishing first Chern class admits a harmonic metric iff it is polystable.

$\boxed{CD + HS = NAH}$

$\boxed{\text{Thm}}$ There is a one-one corresponce between semisimple flat bundles on a curve X and <u>polystable</u> Higgs bundles with vanishing first Chern class on the curve X.

$\boxed{\text{Rem}}$ Irreducible flat bundles are in one-one correspondence with stable Higgs bundles with vanishing first Chern class. Taking the direct sum gives the NAH theorem.

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§5. Digression on Chern connections:

Consider sections s_i and let $K_{ij} = K(s_i, s_j)$

Then ~~d~~ D_K being unitary amounts to saying that

$$dK_{ij} = K(D_K s_i, s_j) + K(s_i, D_K s_j) \quad (1)$$

$$\text{Let } D_K s_i = \underbrace{A_i^m s_m \otimes d\bar{z}}_{(0,1)\text{-part}} + \underbrace{B_i^m s_m \otimes dz}_{(1,0)\text{-part}}$$

substitute into unitarity condition (1)

$$dK_{ij} = \cancel{A_i^m K}\ \overline{A_i^m} K_{mj}\, d\bar{z} + B_i^m K_{mj}\, dz + A_j^m K_{im}\, dz + \overline{B_j^m} K_{im}\, d\bar{z}$$

So,

$$\left.\begin{aligned} \frac{\partial K_{ij}}{\partial \bar{z}} &= \overline{A_i^m} K_{mj} + \overline{B_j^m} K_{im} \\ \frac{\partial K_{ij}}{\partial z} &= A_j^m K_{im} + B_i^m K_{mj} \end{aligned}\right\} \quad (2)$$

Given A and K, we can solve for B conversely given B and K, we can solve for A.

(End of digression)

Example:

Let X be a compact complex curve of genus $g > 1$. Consider the canonical bundle $\mathcal{K}$ of holomorphic 1-forms. Since $c_1(\mathcal{K}) = 2g - 2 \equiv 0 \pmod 2$, we can take "square roots". There are 2^{2g} inequiv. choices. Choose one and denote it $\mathcal{K}^{1/2}$.

Let $E = \mathcal{K}^{1/2} \oplus \mathcal{K}^{-1/2}$. So, $rk(E) = 2$. Also, $c_1(E) = 0$. (can be shown using the Chern polynomial.)

Let dz be a "preferred" (local) section of $\mathcal{K}$ so that ~~$\bar\partial_E$~~ $\bar{\partial}_{\mathcal{K}}(dz) = 0$.

Then take the Higgs field to be

$$\phi := \phi_z\, dz = \begin{pmatrix} 0 & 0 \\ dz^{-1} & 0 \end{pmatrix} dz = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

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So, ϕ sends $\underset{dz^{1/2}\,\in}{\mathcal{K}^{1/2}}$ to $\underset{dz^{-1/2}\,\in}{\mathcal{K}^{-1/2}}$ and $\mathcal{K}^{-1/2}$ to 0

$\mathcal{K}^{-1/2}$ is the only ϕ-invariant subbundle and $c_1(\mathcal{K}^{-1/2}) = 1-g < 0$. So the Higgs bdl is stable and moreover has vanishing Chern class.

Let $h\,dz\,d\bar{z}$ be a Hermitian metric on $\mathcal{K}^{-1}$ (the tangent bundle of X).

minus, not bar

Candidate harmonic metric $K = \begin{pmatrix} h^{-1/2} dz^{-1/2} d\bar{z}^{-1/2} & 0 \\ 0 & h^{1/2} dz^{1/2} d\bar{z}^{1/2} \end{pmatrix}$

transpose conjugate

$$\bar{\Phi} = K^{-1} \phi_z^{\dagger} K \, d\bar{z} = \begin{bmatrix} 0 & h\,dz \\ 0 & 0 \end{bmatrix} = \begin{pmatrix} 0 & h\,dz\,d\bar{z} \\ 0 & 0 \end{pmatrix}$$

$$[\phi_z, \bar{\Phi}_{\bar{z}}] = \begin{pmatrix} -h & 0 \\ 0 & h \end{pmatrix}$$

So, $[\phi, \bar{\Phi}] = \begin{pmatrix} -h & 0 \\ 0 & -h \end{pmatrix} dz \wedge d\bar{z}$

We will now compute the Chern connection. Denote the subbdl $\mathcal{K}^{1/2}$ with index "+" and $\mathcal{K}^{-1/2}$ with index "−".

$\cancel{\text{So, } K_{++} = h^{-1/2} d\bar{z}^{-1/2} d\bar{z}^{-1/2} \; h^{-1/2} dz^{-1/2} d\bar{z}^{-1/2}}$

$\cancel{K_{--} = h^{1/2} dz^{1/2} d\bar{z}^{1/2}, \; K_{+-} = K_{-+} = 0}$

$\cancel{\text{So, } K_{++} = [illegible] \; K}$

So, $K_{++} = h^{-1/2}$, $K_{--} = h^{1/2}$, $K_{+-} = K_{-+} = 0$.

$A_j^m = 0$ (in the Chern connection, i.e. eqn (2))

So, $\dfrac{\partial h^{-1/2}}{\partial z} = B_+^+ h^{-1/2}$, $\dfrac{\partial h^{1/2}}{\partial z} = B_-^- h^{1/2}$

$$B_+^+ = h^{1/2} \frac{\partial h^{-1/2}}{\partial z} = -\frac{1}{2} h^{-1} \frac{\partial h}{\partial z} \quad \cancel{(\text{i.e. } -\tfrac{1}{2} h^{-1} \partial h)}$$

rest of B zero

$$B_-^- = h^{-1/2} \frac{\partial h^{1/2}}{\partial z} = \frac{1}{2} h^{-1} \frac{\partial h}{\partial z} \quad \cancel{(\text{i.e. } \tfrac{1}{2} h^{-1} \partial h / \partial z)}$$

So, Chern connection is $d_E = \begin{pmatrix} -\frac{1}{2} h^{-1} \partial h & \\ & \frac{1}{2} h^{-1} \partial h \end{pmatrix}$

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$$d_E^2 = \frac{1}{2}\begin{pmatrix} -\bar\partial(h^{-1}\partial h) & 0 \\ 0 & \bar\partial(h^{-1}\partial h) \end{pmatrix}$$

$d_E^2 + [\Phi, \bar\Phi] = 0$ (Hitchin's eqn)

so we get one eqn i.e. $\frac{1}{2}\bar\partial(h^{-1}\partial h) + h\, dz\wedge d\bar z = 0$

i.e. $\frac{1}{2}(\bar\partial h^{-1}\wedge\partial h + h^{-1}\bar\partial\partial h) + h\, dz\wedge d\bar z = 0$

i.e. $\frac{\partial h^{-1}}{\partial \bar z}\frac{\partial h}{\partial z}\, d\bar z\wedge dz + h^{-1}\frac{\partial^2 h}{\partial\bar z\partial z}\, d\bar z\wedge dz$
$= -2h\, dz\wedge d\bar z$

i.e. $\left(\frac{\partial h}{\partial z}\right)\left(\frac{\partial h^{-1}}{\partial \bar z}\right) + h^{-1}\frac{\partial^2 h}{\partial z\partial\bar z} = 2h$

This is a ~~di~~ PDE in h. Solving it gives the harmonic metric.

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Ref.

"Canonical" references:

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[3] N. J. Hitchin. The self-duality equations on a Riemann surface. Proceedings of the London Mathematical Society, s3-55(1).

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Other ~~to~~ references:

[5] Hiro Tanaka. A baby version of the nonabelian Hodge theorem.

[6] Robert Maschal. Introduction to Higgs bundles

[7] Richard Wentworth. Higgs bundles and local systems on Riemann surfaces.

[8] Alberto Garcia-Raboso and Steven Rayan. Introduction to nonabelian Hodge theory.

[9] Andrew Sanders. Harmonic maps—from representations to Higgs bundles.

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