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HYPERKÄHLER MANIFOLDS

by Nigel HITCHIN

1. INTRODUCTION

1.1. "I then and there felt the galvanic circuit close; and the sparks which fell from it were the fundamental equations between i, j and k exactly as I have used them ever since" [H].

Hamilton's conviction that the quaternions should play as fundamental a rôle as the complex numbers in mathematics and physics was never realized in his day. There exists now, however, a rich theory of manifolds based on the algebra of quaternions which goes some way towards vindicating his belief. These manifolds moreover arise naturally within the context of the equations of mathematical physics. They are the *hyperkähler manifolds.*

The "fundamental equations between i, j and k" which, on that October day in 1843, Hamilton carved with such enthusiasm on Brougham Bridge were of course

$$i^2 = j^2 = k^2 = ijk = -1$$

A hyperkähler manifold is a manifold (necessarily of dimension a multiple of four) which admits an action on tangent vectors of the same i, j and k in a manner which is compatible with a metric. More precisely,

DEFINITION. *A hyperkähler manifold is a Riemannian manifold with three covariant constant orthogonal automorphisms I, J and K of the tangent bundle which satisfy the quaternionic identities $I^2 = J^2 = K^2 = IJK = -1$*

1.2. Recall that a Riemannian manifold which has just one such automorphism is called a Kähler manifold. The name "hyperkähler", which originated with E. Calabi [Ca], is a fair description – the metric is Kählerian for several complex stuctures – even though it does recall Grassmann's "hypercomplex numbers" rather than Hamilton's quaternions. There is, however, an essential difference between Kähler and hyperkähler manifolds. A Kähler metric on a given complex manifold can be modified to another one simply by adding a hermitian form $\partial\bar{\partial}f$ for an arbitrary sufficiently small C^∞ function f. Thus the space of Kähler metrics is infinite dimensional. It is moreover easy to find examples of Kähler manifolds. Any complex submanifold of $\mathbf{C}P^n$ inherits a Kähler metric and so simply writing down algebraic equations for a projective variety gives a vast number of examples.

By contrast, hyperkähler metrics are much more rigid. On a compact manifold, if one such metric exists, then up to isometry there is only a finite dimensional space of them. Nor is it easy to find examples. Certainly we will never find them as quaternionic submanifolds of the quaternionic projective space $\mathbf{H}P^n$ [Gr].

1.3. The concept of a hyperkähler manifold arose first in 1955 with M. Berger's classification of the holonomy groups of Riemannian manifolds. On a hyperkähler manifold, parallel translation preserves I, J and K (since they are covariant constant) and so the holonomy group is contained in both the orthogonal group $O(4n)$ and the group $GL(n, \mathbf{H})$ of quaternionic invertible matrices (i.e. those linear transformations which commute with right multiplication by i, j and k). The maximal such intersection is $Sp(n)$, the group of $n \times n$ quaternionic unitary matrices. This group appeared in Berger's list.

1.4. The group $Sp(n)$ is also an intersection of $U(2n)$ and $Sp(2n, \mathbf{C})$, the linear transformations of $\mathbf{C}^{2n}$ which preserve a non-degenerate skew form. Thus a hyperkähler manifold is naturally a complex manifold with a holomorphic symplectic form. One can see this explicitly by taking the three Kähler two-forms

$$\omega_1(X,Y) = g(IX,Y) \qquad \omega_2(X,Y) = g(JX,Y) \qquad \omega_3(X,Y) = g(KX,Y)$$

defined for the complex structures I, J and K. With respect to the complex structure I, the complex form $\omega_c = \omega_2 + i\omega_3$ is non-degenerate and covariant constant, hence closed and holomorphic.

This point of view provides guidance in the search for examples of hyperkähler manifolds, and elucidates the sort of differential equation which needs to be solved. In the first place the holomorphic volume form ${\omega_c}^n$ must for a hyperkähler manifold give a covariant constant trivialization of the canonical line bundle. The curvature of this bundle for any Kähler metric is the Ricci form and so a hyperkähler metric has in particular vanishing Ricci tensor. In the lowest dimension – four – this means that such metrics satisfy the Riemannian version of the Einstein vacuum equations.

Given a *compact* Kähler manifold with holomorphically trivial canonical bundle, the Calabi-Yau theorem [Y] provides the existence of a Kähler metric with vanishing Ricci tensor. Furthermore, a much older theorem due to S. Bochner [Bo] shows that any holomorphic form on a compact Kähler manifold with zero Ricci tensor is covariant constant. Thus for every compact Kähler manifold with a holomorphic symplectic form, an application of these two theorems yields a hyperkähler metric on the same manifold. This satisfactory state of affairs can be used to prove the existence of hyperkähler metrics on many examples of complex manifolds. The most fundamental is the K3 surface – the only non-trivial example in 4 real dimensions. In higher dimensions, the Hilbert scheme of zero cycles on a K3-surface or a 2-dimensional complex torus yields a natural class of holomorphic symplectic manifolds and hence hyperkähler metrics [Bea].

1.5. In this exposition, however, we shall seek something more than existence. We should like to *construct* solutions in a more explicit manner, in order to gain a better understanding of hyperkähler manifolds and to experience the richness of their geometry. The ability to do this is a relatively recent phenomenon. Indeed, twenty years ago it was hardly possible to write down any non-trivial Riemannian metric with zero Ricci tensor.

There are two main routes to constructing hyperkähler metrics: (a) twistor theory, (b) hyperkähler quotients.

The twistor approach is based on R. Penrose's original work in relativity [P]. It provides an *encoding* of the data for such a metric in terms of holomorphic geometry. One might say that the differential equations are reduced to just one – the Cauchy-Riemann equation. Breaking that code in order to write down the metric is sometimes a difficult task. Deriving global properties of the metric such as completeness is almost impossible. On the other hand, the quotient construction yields this sort of property quite easily, even if it is not as general as the twistor method.

The hyperkähler quotient construction arose also out of questions of mathematical physics [HKLR], in this case supersymmetry. In practical terms there are two ways of using it. The first is a finite-dimensional construction, whereby determining the actual metric involves solving algebraic equations. The second involves the use of the method in infinite dimensions, even though the quotient itself may be finite-dimensional. Here, one needs to solve differential equations to find the metric. They are, however, equations for which in many cases methods of solution are known so that we have *in principle* more information than an existence theorem.

1.6. We shall illustrate these constructions by a representative collection of examples which are chosen according to our guiding principle of seeking complex symplectic manifolds. These hyperkähler manifolds are all *a priori* complex manifolds with holomorphic symplectic forms:

(i) resolutions of rational surface singularities

(ii) coadjoint orbits of complex Lie groups

(iii) spaces of representations of a surface group in a complex Lie group

(iv) the space of based rational maps $f : \mathbf{C}P^1 \to \mathbf{C}P^1$ of degree k

(v) the space of based loops in a complex Lie group

The construction of hyperkähler metrics on these spaces is contained in the work of P. B. Kronheimer, S. K. Donaldson, and others. What is perhaps remarkable is that these diverse spaces nearly all inherit a hyperkähler metric through special cases of solutions to the anti-self-dual Yang-Mills equations in $\mathbf{R}^4$. Those physically motivated equations themselves are ultimately based on the identification of $\mathbf{R}^4$ with the quaternions. Hamilton's ghost may yet rest content.

2. THE TWISTOR CONSTRUCTION

2.1. A hyperkähler manifold M^{4n} is, by definition, endowed with three complex structures I, J and K. In fact, if $\mathbf{u} = (a, b, c) \in \mathbf{R}^3$ then

$$(aI + bJ + cK)^2 = -(a^2 + b^2 + c^2)$$

so that if $\|\mathbf{u}\| = 1$ we have another covariant constant (and hence integrable) complex structure $I_{\mathbf{u}}$. The hyperkähler metric is Kählerian with respect to all of these structures. The *twistor space* of M is the product $Z = M \times S^2$. The tangent space to the 2-sphere S^2 has a natural complex structure I_0 (considering it as the Riemann sphere) and for $X \in T_m M$, $Y \in T_{\mathbf{u}} S^2$ we put

$$\mathbf{I}(X, Y) = (I_{\mathbf{u}} X, I_0 Y)$$

which defines a complex structure on the tangent space $T_m M \oplus T_{\mathbf{u}} S^2$ to the twistor space Z. It is a theorem [HKLR],[S],[AHS] that this almost complex structure is *integrable* and so Z is a complex manifold of complex dimension $2n + 1$.

2.2. The twistor space has the following features. Firstly, the projection onto $S^2 \cong \mathbf{C}P^1$ is holomorphic. Secondly, the antipodal map σ on the unit sphere takes $I_{\mathbf{u}}$ to $-I_{\mathbf{u}}$ and I_0 to $-I_0$, so we may consider Z as a *real* complex manifold (a complex manifold with an antiholomorphic involution – like a projective variety defined by equations with real coefficients).

The triple of Kähler forms ω_1, ω_2 and ω_3 on M is determined by the choice of an orthonormal frame in $\mathbf{R}^3$. The complex two-form ω_c can then be defined for all complex structures $I_{\mathbf{u}}$ lifted to $M \times SO(3)$. When we obtain S^2 by dividing $SO(3)$ by a circle action, ω_c is no longer a true two-form, but is twisted by the tangent line bundle $\mathcal{O}(2)$ of $\mathbf{C}P^1$. It is holomorphic.

Finally, each point $m \in M$ defines a section $\{m\} \times S^2$ of the projection $p : Z = M \times S^2 \to \mathbf{C}P^1$ which is both holomorphic and real (i.e. σ-invariant). The normal bundle of this section is isomorphic as a holomorphic bundle to $\mathbf{C}^{2n} \otimes \mathcal{O}(1)$. We then have the basic theorem [HKLR]

THEOREM 1. *Let M^{4n} be a hyperkähler manifold and Z its twistor space. Then*

(1) Z is a holomorphic fibre bundle $p^Z \rightarrow \mathbf{C}P^1$ over the projective line

(2) the bundle admits a family of holomorphic sections, each with normal bundle isomorphic to $\mathbf{C}^{2n} \otimes \mathcal{O}(1)$

(3) there exists a holomorphic section ω of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$ defining a symplectic form on each fibre

(4) Z has a real structure σ compatible with (1), (2) and (3) and inducing the antipodal map on $\mathbf{C}P^1$.

Conversely, the parameter space of real sections of any complex manifold Z^{2n+1} satisfying (1) to (4) is a $4n$-dimensional manifold with a hyperkähler metric for which Z is the twistor space.

(Here $\mathcal{O}(k)$ denotes the pull-back by p of the corresponding line bundle on $\mathbf{C}P^1$ and T_F^* the cotangent bundle along the fibres of p. From now on, for a vector bundle E, $E(k)$ will denote the tensor product $E \otimes \mathcal{O}(k)$).

The key to the converse is the fact that the tangent space to the space of sections is

$$H^0(\mathbf{C}P^1; T_F) \cong H^0(\mathbf{C}P^1; T_F(-1)) \otimes H^0(\mathbf{C}P^1; \mathcal{O}(1))$$

and the skew form ω gives the first factor a symplectic form and the Wronskian the second, providing a symmetric form – a metric – on the tensor product.

2.3. This is the "encoding" via the Penrose twistor space of a hyperkähler metric. The differential equations are all contained in the non-linearity of the geometrical construction of a suitable Z. Moreover, Z itself can be considered as a natural object within the realm of holomorphic symplectic geometry. Formally speaking, one may consider the projection p and the structure of the form ω as giving a symplectic manifold defined over the field of rational functions in one variable. The sections are then rational points on this variety. In particular examples, this "Diophantine" aspect is evident, as finding the sections results in solving algebraic equations with polynomials [H1],[H4].

2.4. Example. Let Z be the total space of the rank 2 vector bundle $E = \mathbf{C}^2(1)$ over $\mathbf{C}P^1$. This is the twistor space for the flat hyperkähler structure on $\mathbf{R}^4$.

Now let s be a non-vanishing section of E, then s generates a trivial sub-line bundle with quotient $\mathcal{O}(2)$. Thus translation by a multiple of s describes E as a principal $\mathbf{C}$-bundle over the total space T of $\mathcal{O}(2)$. Given $\lambda \in \mathbf{R}$ we define a line bundle L^λ over T by

$$L^\lambda = E \times_{\mathbf{C}} \mathbf{C}$$

where $u \in \mathbf{C}$ acts by $w \mapsto e^{\lambda u} w$

If z denotes the tautological section of the pull-back of $\mathcal{O}(2)$ to its total space T, then

$$Z = \{(x, y) \in L^\lambda(1) \oplus L^{-\lambda}(1) : xy = z\}$$

defines a twistor space for a complete hyperkähler metric on $\mathbf{R}^4$. This is the *Taub-NUT* metric. A derivation of the explicit metric from the twistor construction can be found in [Be].

3. THE HYPERKÄHLER QUOTIENT

3.1 The twistor construction for hyperkähler metrics has as its starting point the complex structures I, J and K. In fact, it is natural to consider such structures alone on manifolds, without the existence of a compatible metric. This is the more general theory of *hypercomplex manifolds* and the corresponding twistor theory simply involves deleting condition (3) in Theorem 1.

The quotient construction, by contrast, emphasizes not the complex structures but instead the corresponding Kähler forms ω_1, ω_2 and ω_3. In this case, by contraction with the inverse forms on cotangent vectors, we can recover I, J and K and the metric itself. Put another way, $Sp(n)$ is the stabilizer of ω_1, ω_2, ω_3 whereas $GL(n, \mathbf{H})$ is the stabilizer of I, J, K.

A hyperkähler manifold can be characterized in a very straightforward manner using these forms [HKLR]:

THEOREM 2. *Let M^{4n} be a manifold with 2-forms ω_1, ω_2, ω_3 whose stabilizer in $GL(4n, \mathbf{R})$ at each point $m \in M$ is conjugate to $Sp(n)$. Then the forms define a hyperkähler structure if and only if they are closed.*

This theorem, which is a straightforward consequence of the Newlander-Nirenberg theorem, places the theory of hyperkähler manifolds firmly within the context of symplectic geometry.

3.2. The hyperkähler quotient is modelled on the Marsden-Weinstein quotient construction in symplectic geometry.

Recall that if (M, ω) is a symplectic manifold with a symplectic action of a Lie group G, then under mild assumptions one can define an equivariant moment map $\mu : M \to \mathfrak{g}^*$ taking values in the dual of the Lie algebra of G: for each $\xi \in \mathfrak{g}$ the vector field X_ξ generated by the action satisfies $d\mu(\xi) = i(X_\xi)\omega$. The moment map is ambiguous up to the addition of a constant $\zeta \in \mathfrak{z} \subseteq \mathfrak{g}^*$ where $\mathfrak{z}$ is the space of G-invariant elements of $\mathfrak{g}^*$.

The symplectic quotient construction consists of choosing a regular value $\zeta \in \mathfrak{z}$ for μ and then the form ω restricted to the submanifold $\mu^{-1}(\zeta)$ is invariant and degenerate in the directions of the G-orbits and hence descends to a form $\bar{\omega}$ on the quotient manifold $\mu^{-1}(\zeta)/G$. The form $\bar{\omega}$ is symplectic.

3.3. Suppose now that M is a hyperkähler manifold, with a Lie group G acting so as to preserve the three Kähler forms ω_1, ω_2 and ω_3. We obtain three moment maps μ_1, μ_2 and μ_3 or equivalently a vector-valued moment map

$$\mu : M \to \mathfrak{g}^* \otimes \mathbf{R}^3$$

We then have [HKLR]

THEOREM 3. *If $\zeta \in \mathfrak{z} \otimes \mathbf{R}^3$ is a regular value for the hyperkähler moment map μ, then $\mu^{-1}(\zeta)/G$ is a hyperkähler manifold.*

The proof, using Theorem 2, is direct. Each form ω_i descends to a form $\bar{\omega}_i$ just as in the symplectic case. What remains to be checked is that the quaternionic algebraic relations between $\bar{\omega}_1$, $\bar{\omega}_2$ and $\bar{\omega}_3$ are still satisfied.

Note that

$$\dim \mu^{-1}(\zeta)/G = \dim M - 4 \dim G$$

3.4. The symplectic quotient itself has an important rôle to play in *Kähler geometry*, for if M is a Kähler manifold and G preserves the complex structure as well as the symplectic form, then the symplectic quotient is again Kählerian [Ki],[HKLR]. The complex structure of the quotient is then identified with the complex quotient M^s/G^c of a certain open set M^s of *stable* points in M by a complex group action which holomorphically extends that of G. A point is stable if its G^c-orbit intersects $\mu^{-1}(\zeta)$. This symplectic point of view works very well for projective varieties and correlates effectively with geometric invariant theory [Ki].

3.5. Example. A simple illustration of the Kähler symplectic quotient is the following. Take $M = \mathbf{C}^n$ with its standard hermitian structure and the action of the circle $G = S^1$ by scalar multiplication. The moment map $\mu : \mathbf{C}^n \to i\mathbf{R}$ is

$$\mu(z) = i\|z\|^2$$

and $\mu^{-1}(i) = S^{2n-1}$. The symplectic quotient is therefore

$$S^{2n-1}/S^1 = \mathbf{C}P^{n-1} = \mathbf{C}^n \backslash \{0\}/\mathbf{C}^*$$

so that $M^s = \mathbf{C}^n \backslash \{0\}$ and $\mathbf{C}P^{n-1}$ inherits a natural Kähler metric – the *Fubini-Study* metric.

3.6. In the hyperkähler situation, since the forms ω_1, ω_2 and ω_3 define I, J and K then if an action of G preserves the symplectic forms, it automatically preserves the complex structures. Fixing attention on one such complex structure I, the function $\mu_c = \mu_2 + i\mu_3$ is actually holomorphic. It is the moment map with respect to the holomorphic symplectic form ω_c of the action of the complex group G^c. Thus $\mu^{-1}(\zeta)$ can be rewritten in the form

$$\mu_1{}^{-1}(a) \cap \mu_c{}^{-1}(\alpha)$$

for some $a \in \mathfrak{z}$ and $\alpha \in \mathfrak{z} \otimes \mathbf{C}$.

From this point of view, the hyperkähler quotient is the symplectic quotient of the Kähler manifold ${\mu_c}^{-1}(\alpha)$. It follows, that when due account is taken of stability, the induced complex structure I on the hyperkähler quotient is simply that of ${\mu_c}^{-1}(\alpha)/G^c$ – the holomorphic version of the symplectic quotient.

3.7. As the complex structure I varies (equivalently as we look at the different fibres of the twistor space $p : Z \to \mathbf{C}P^1$) the twistor space of the hyperkähler quotient is now essentially the fibre-wise symplectic quotient of Z by the holomorphic action of G^c. For each complex structure, the complex moment map μ_c is the restriction of a holomorphic section $\tilde{\mu}$ of $\mathfrak{g}^c(2)$, incorporating the twist of the form ω of Theorem 1. The twistor space of the quotient is then $\tilde{\mu}^{-1}(\zeta)^s/G^c$ for $\zeta \in \mathfrak{g} \otimes \mathbf{R}^3 \subset \mathfrak{g}^c \otimes H^0(\mathbf{C}P^1; \mathcal{O}(2)) \subset H^0(Z; \mathfrak{g}^c(2))$.

In the context of (2.3) it is just the symplectic quotient over the field of rational functions.

3.8. In practice it is remarkable that one may obtain interesting examples by starting with the flat hyperkähler manifold $M^{4n} = \mathbf{H}^n$ and taking a quotient by a linear action of a group. The closest analogue of the projective space construction in (3.5) is the following example.

Let V be an n-dimensional hermitian vector space and V^* its dual. Then

$$M^{4n} = V \oplus V^*$$

is a flat hyperkähler manifold. Let $G = S^1$ and let $u \in S^1$ act on M by

$$u(x, \xi) = (ux, u^{-1}\xi)$$

This action preserves the hyperkähler 2-forms and using the complex and real moment maps as in (3.6) one finds

$$\mu_c(x, \xi) = \xi(x) \qquad \mu_1(x, \xi) = i(\|x\|^2 - \|\xi\|^2)$$

Thus if $\zeta = (i, 0, 0) \in \mathfrak{g}^* \otimes \mathbf{R}^3$ then

$$\mu^{-1}(\zeta) = \{(x, \xi) : \xi(x) = 0, \|x\|^2 - \|\xi\|^2 = 1\}$$

The second condition determines the set of stable points M^s as the set where $x \neq 0$ and then $M^s/\mathbf{C}^*$ is the *cotangent bundle* of $\mathbf{P}(V) = \mathbf{C}P^{n-1}$. This, however, is only one of the 2-sphere of complex structures on this hyperkähler manifold. Because of the symmetries in this example, the others can be identified with the quotients $\mu_c^{-1}(\alpha)/\mathbf{C}^*$ for $\alpha \neq 0$ which are affine bundles over $\mathbf{C}P^{n-1}$ with group of translations the cotangent bundle. Note that complex cotangent bundles are naturally complex symplectic manifolds.

This particular metric for $n = 2$ is the *Eguchi-Hanson* metric [EH]. The generic complex structure is that of an affine quadric in $\mathbf{C}^3$. The higher-dimensional examples were first found by E. Calabi [Ca].

3.9. One general class of hyperkähler moment maps we shall encounter is based on the following example.

Let G be a Lie group with a bi-invariant metric, and put

$$M = \mathfrak{g} \otimes \mathbf{H}$$

where $\mathfrak{g}$ is the Lie algebra. This is a flat hyperkähler manifold, and the adjoint action of G preserves the symplectic forms. The three hyperkähler moment maps are then

$$\begin{aligned} \mu_1(A) &= [A_0, A_1] + [A_2, A_3] \\ \mu_2(A) &= [A_0, A_2] + [A_3, A_1] \\ \mu_3(A) &= [A_0, A_3] + [A_1, A_2] \end{aligned}$$

where $A \in \mathfrak{g} \otimes \mathbf{H}$ is defined by $A = A_0 + A_1 i + A_2 j + A_3 k$. Note that the complex moment map is given by

$$\mu_c(A) = [A_0 + iA_1, A_2 + iA_3]$$

and putting $\alpha = A_0 + iA_1$ and $\beta = A_2 + iA_3$ the three moment maps can be written as a complex and real moment map

$$\begin{aligned} \mu_c &= [\alpha, \beta] \\ \mu_r &= [\alpha, \alpha^*] + [\beta, \beta^*] \end{aligned}$$

It is worth pointing out here that replacing the A_i by covariant derivatives ∇_i yields the anti-self dual Yang-Mills equations as the single equation $\mu = 0$. We shall see versions of this result in the examples which follow.

4. RATIONAL DOUBLE POINTS

4.1. In the study of surface singularities, the *rational double points* are characterized by the property that their minimal resolution has trivial canonical bundle – these are thus, locally, 2-dimensional complex manifolds with holomorphic symplectic forms. The basic model for such a singularity is the space $\mathbf{C}^2/\Gamma$ where Γ is a finite subgroup of $SU(2)$. We shall describe here the construction of Kronheimer [K1],[K2] for a hyperkähler metric on the minimal resolution. We have in (3.8) already encountered such a metric – the Eguchi-Hanson metric. This was defined on the cotangent bundle of $\mathbf{C}P^1$, but the zero section of this has self-intersection -2 and so can be blown down to a singularity – the ordinary double point $\mathbf{C}^2/\pm 1$. Kronheimer's construction proceeds (as in (3.8)) by using hyperkähler quotients.

4.2. Let Γ be a finite subgroup of $SU(2)$ and let $V = L^2(\Gamma)$ be the finite-dimensional Hilbert space of functions on Γ, $U(V)$ the unitary group of V and $\mathfrak{u}(V)$ its Lie algebra. Then, as in (3.9), $\mathfrak{u}(V) \otimes \mathbf{H}$ is a flat hyperkähler manifold – a quaternionic vector space with a compatible inner product.

Now, since $SU(2)$ is isomorphic to the unit quaternions, Γ acts on the left on both the quaternions and $L^2(\Gamma) = V$ and so has an action on $\mathfrak{u}(V) \otimes \mathbf{H}$ commuting with right multiplication by quaternions. Thus

$$M = (\mathfrak{u}(V) \otimes \mathbf{H})^\Gamma$$

the space of invariant elements, is also a flat hyperkähler manifold.

The projective group $PU(V)$ acts by conjugation on $\mathfrak{u}(V) \otimes \mathbf{H}$ preserving the hyperkähler structure and so $G = PU(V)^\Gamma$ acts on M in the same way. This is the context in which we may take the hyperkähler quotient.

4.3. To gain more information, we decompose V into irreducible representations. Now

$$V = L^2(\Gamma) = \bigoplus_{i=0}^{l} (V_i^L \otimes V_i^R)$$

where the sum is over the irreducible representations of Γ under left and right action, and V_0 is the trivial representation. Thus

$$U(V)^\Gamma = U(V_0^R) \times \ldots \times U(V_l^R)$$

Hence if $\dim V_i^R = \dim V_i^L = n_i$, then

$$\dim G = \sum_0^l n_i^2 - 1$$

We can also see clearly the centre of G from this decomposition, and from that the subspace $\mathfrak{z} \in \mathfrak{g}^*$ of invariant elements. We have

$$\dim \mathfrak{z} = l$$

4.4. The structure of the irreducible representations of Γ is conveniently encoded through the McKay correspondence [McK]. To each subgroup Γ (cyclic, or binary dihedral, tetrahedral, octahedral or icosahedral) there corresponds a Dynkin diagram of type A_l, D_l, E_6, E_7, E_8 respectively. Each vertex of the diagram corresponds to a non-trivial irreducible representation of Γ.

In Lie algebra theory, each vertex corresponds to a simple root α_i. The highest root is expressed in terms of the simple roots by

$$\sum_{i=1}^{l} n_i \alpha_i$$

From McKay the integer n_i is the dimension of the representation space corresponding to the ith vertex. The whole situation is simplified by introducing the *extended* Dynkin diagram incorporating the highest root. The relation

$$\sum_{i=0}^{l} n_i \alpha_i = 0$$

with $n_0 = 1$ and α_0 the negative of the highest root then puts the trivial representation on the same footing as the others.

4.5. To find the dimension of M in (4.2) requires a knowledge of the relationship between the tensor product of the defining 2-dimensional representation of $\Gamma \subset SU(2)$ and the other representations V_i. This again is provided by the McKay correspondence:

$$\mathbf{C}^2 \otimes V_i = \bigoplus_j a_{ij} V_j$$

where $a_{ij} = 1$ if the vertices of the extended Dynkin diagram are adjacent and zero otherwise.

As a complex vector space, $M = (\mathrm{Hom}(V,V) \otimes \mathbf{C}^2)^\Gamma$ and so

$$\begin{aligned}\dim_{\mathbf{C}} M &= \dim \bigoplus_{i,j} (\mathrm{Hom}(V_j^L \otimes V_j^R, V_i^L \otimes V_i^R \otimes \mathbf{C}^2))^\Gamma \\ &= \dim \bigoplus_{i,j} a_{ij} \mathrm{Hom}(V_i^L, V_j^L) \\ &= \sum_{i,j} a_{ij} n_i n_j\end{aligned}$$

But now applying the Cartan form to the relation in (4.4) gives

$$-2 \sum_0^l n_i^2 + \sum_{i,j} a_{ij} n_i n_j = 0$$

and so

$$\dim_{\mathbf{R}} M = 4 \sum_0^l n_i^2 = 4(\dim G + 1)$$

from (4.3).

Thus, provided G is shown to act freely, we have a hyperkähler quotient of dimension $\dim M - 4\dim G = 4$. Note that since $\mathfrak{z}$ is l-dimensional there is a choice in moment map, which must be suitably exercised to obtain a free action.

4.6. The relationship with the resolution of $\mathbf{C}^2/\Gamma$ may be seen by considering $\zeta = (\zeta_1, 0, 0)$. Since we are essentially in the situation of (3.9), the hyperkähler moment map equations can be written as equations for a pair of complex matrices

$$[\alpha, \beta] = 0$$
$$[\alpha, \alpha^*] + [\beta, \beta^*] = \zeta_1$$

but where α,$\beta \in \mathrm{Hom}(V, V)$ define a Γ-invariant element of $\mathrm{Hom}(V, V) \otimes \mathbf{C}^2$.

This invariance means that if $e \in V$ is a common eigenvector of α and β (which exists since they commute) with eigenvalues a, b then $(\alpha, \beta)e = (a, b)e$ and

$$(\alpha, \beta)e^\gamma = (a, b)^\gamma e^\gamma$$

where $(a, b) \mapsto (a, b)^\gamma$ is the defining action of Γ on $\mathbf{C}^2$.

Since Γ acts freely on $\mathbf{C}^2\backslash\{0\} = \mathbf{H}\backslash\{0\}$ (multiplication by a non-zero quaternion) if one eigenvalue pair is non-zero then we have $|\Gamma| = \dim V$ distinct eigenvalue pairs. In particular, the Γ-orbit of one such pair (a, b) is well-defined by (α, β). This provides a map p, holomorphic with respect to I, to $\mathbf{C}^2/\Gamma$. If ζ_1 is chosen appropriately, the quotient is non-singular, so that once one proves that p is biholomorphic outside the origin in $\mathbf{C}^2/\Gamma$ the map is a resolution.

The four-dimensional metrics produced this way are not only complete hyperkähler manifolds, but they are also *asymptotically locally Euclidean (ALE)* meaning that they approach rapidly the Euclidean metric on $\mathbf{R}^4/\Gamma$ at infinity.

5. COADJOINT ORBITS

5.1. For any Lie group, an orbit in the dual of the Lie algebra is a symplectic manifold. This is the canonical Kostant-Kirillov symplectic structure. If the group G is compact, then the orbits are Kähler manifolds. This fact may be exploited to obtain all the irreducible representations of G within the context of geometric quantization (the Borel-Weil theorem).

The existence of a symplectic structure is completely general. In particular, a coadjoint orbit of a complex Lie group G^c is a holomorphic symplectic manifold. These manifolds in many cases possess natural *hyperkähler* metrics due to the work of Kronheimer [K3]. The Eguchi-Hanson metric already provides us with one example – the affine quadric in $\mathbf{C}^3$ (which is the generic complex structure) is the orbit under $SL(2,\mathbf{C})$ of a semi-simple element in $\mathfrak{g} = \mathfrak{g}^*$. We shall give here the general construction for a *regular* semisimple orbit – one of the form G^c/T^c where G^c is a complex semisimple Lie group and T^c a maximal complex torus. The compact analogue of this is the flag manifold G/T, the orbit type which features prominently in the Borel-Weil theorem.

5.2. The metric is produced by an application of the hyperkähler quotient construction to an infinite-dimensional flat hyperkähler manifold. The setting is that of a special case of the anti-self-dual Yang-Mills equations. We consider a compact semisimple group G and a trivial principal G-bundle P over $\mathbf{R}^4\backslash\{0\}$, and the space of connections on P. This is an infinite-dimensional affine space modelled on the vector space $\Omega^1(\mathbf{R}^4\backslash\{0\}; \mathrm{ad}P)$ of 1-forms with values in the Lie algebra bundle. This is itself a quaternionic vector space, inheriting its structure from the identification of $\mathbf{R}^4$ with $\mathbf{H}$. The left action of the unit quaternions $Sp(1)$ commutes with the right action and it follows that the space of $Sp(1)$-invariant connections is also quaternionic. The invariant automorphisms or *gauge transformations* act on this space.

To define a metric requires a closer attention to boundary conditions, but having done that, one may consider the hyperkähler quotient of this affine space by the group of gauge transformations to obtain a hyperkähler quotient.

5.3. The $Sp(1)$-invariance condition above throws the emphasis onto a radial variable $s \in (-\infty, 0]$. The moment map equations can then be transformed to the following non-linear system of equations

$$[\frac{d}{ds} + B_0, B_1] + [B_2, B_3] = 0$$
$$[\frac{d}{ds} + B_0, B_2] + [B_3, B_1] = 0$$
$$[\frac{d}{ds} + B_0, B_3] + [B_1, B_2] = 0$$

for functions $B_i(s)$ with values in the Lie algebra $\mathfrak{g}$. Note how these equations compare with those in (3.9) putting $A_0 = d/ds + B_0$ and $A_i = B_i$ for $i > 0$.

The boundary conditions which give rise to this moment map are defined by comparison with a particular configuration given by $B_0 = 0$ and $B_i = \tau_i$ where τ_1, τ_2, τ_3 lie in a fixed Cartan subalgebra $\mathfrak{h}$. They are to be chosen such that their common centralizer is $\mathfrak{h}$ itself. This is clearly a solution to the equations. The space of operators $d/ds + B_0, B_1, B_2, B_3$ which are close to this model configuration in some exponentially-weighted C^1 norm [K3] then admits a well-defined inner product. (In the 4-dimensional interpretation above this is simply the L^2 inner product).

The adjoint action of the group of smooth functions $g(s)$ with values in G on the four operators $d/ds + B_0, B_1, B_2, B_3$ then gives a hyperkähler group action. (This is the group of invariant gauge transformations in the 4-dimensional formalism).

It is easy to see in this case that the quotient is finite-dimensional, since by a gauge transformation the operator $d/ds + B_0$ can be transformed to d/ds leaving an ordinary differential equation in B_1, B_2 and B_3 with equivalence under the finite-dimensional group G.

5.4. Identifying the complex structures on the quotient involves an extra theorem. Here one chooses the complex structure and rewrites the moment map equations as in (3.9) with a real and complex part

$$\frac{d\beta}{ds} + [\alpha, \beta] = 0$$
$$\frac{d}{ds}(\alpha + \alpha^*) + [\alpha, \alpha^*] + [\beta, \beta^*] = 0$$

where $\alpha = (B_0 + iB_1)(s)$ and $\beta = (B_2 + iB_3)(s)$.

One then shows that if the boundary conditions are satisfied, the map

$$(\alpha, \beta) \mapsto \beta(0) \in \mathfrak{g}^c$$

identifies the space of equvalence classes of solutions to the above equations with the adjoint orbit of $\tau_2 + i\tau_3$. For G^c semisimple this is isomorphic to the coadjoint orbit. The proof itself is modelled on a theorem of Donaldson [D1].

5.5. Notice that there is a choice in the moment map reflected this time in the boundary conditions. Part of that choice involves $\tau_2 + i\tau_3$, the particular coadjoint orbit, but the extra choice of τ_1 in the Cartan subalgebra gives a family of hyperkähler metrics.

We may remark also that although finding the metric explicitly involves solving the non-linear equations in (5.3), they are a form of *Nahm's equations*, the general solution of which can be described in terms of the geometry of the Jacobian of an algebraic curve [H3]. There is clearly much more than an existence theorem involved here.

6. REPRESENTATIONS OF SURFACE GROUPS

6.1. If Σ is a compact oriented surface, its fundamental group has an intrinsically symplectic nature [G]. In particular, for any Lie group G with an invariant inner product on its Lie algebra, the moduli space of irreducible representations of $\pi_1(\Sigma)$ is a symplectic manifold. This is the space $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\Sigma), G)/G$ where G acts by conjugation. The tangent space at a representation can be identified with the cohomology group $H^1(\pi_1(\Sigma); \mathfrak{g})$ and the bilinear form on $\mathfrak{g}$ gives a skew pairing to $H^2(\pi_1(\Sigma))$ which is generated by a fundamental class.

When G is a compact group, a choice of complex structure on Σ makes the moduli space into a Kähler manifold, its holomorphic structure being that of the space of stable holomorphic bundles on the Riemann surface Σ. This is the theorem of Narasimhan and Seshadri [NS].

For a *complex* semi-simple group G^c, the moduli space

$$\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\Sigma), G^c)/G^c$$

is naturally a complex symplectic manifold and has in fact a natural hyperkähler metric, also determined by a choice of complex structure on Σ.

6.2. The context for this metric is again an infinite-dimensional hyperkähler quotient construction.

Let $\mathcal{A}^c$ be the affine space of all G^c-connections on a principal G^c-bundle over Σ which we assume has a fixed reduction to the maximal compact group G. This provides a metric on any associated vector bundle, and a conjugation operation on the bundle $\mathrm{ad}P$ associated to the adjoint representation of G. Each tangent vector to $\mathcal{A}^c$ may be considered as a 1-form $\alpha \in \Omega^1(\Sigma; \mathrm{ad}P \otimes \mathbf{C})$ and if $\mathrm{Tr}(AB)$ denotes the inner product on $\mathfrak{g}$, we have a complex symplectic form defined by

$$\omega(\alpha, \beta) = \int_\Sigma \mathrm{Tr}(\alpha \wedge \beta)$$

Given a complex structure on Σ, we may decompose $\alpha = \alpha^{1,0} + \alpha^{0,1}$ into forms of different type and then there is a real symplectic form – the Kähler form of the metric

$$\|\alpha\|^2 = \int_\Sigma \mathrm{Tr}(\alpha^{1,0} \wedge \alpha^{1,0*}) - \mathrm{Tr}(\alpha^{0,1} \wedge \alpha^{0,1*})$$

This makes $\mathcal{A}^c$ into an infinite-dimensional hyperkähler manifold. The group $\mathcal{G}$ of gauge transformations acts on $\mathcal{A}^c$ preserving this structure.

6.3. The moment maps for this hyperkähler action are expressed in terms of curvature. This is a consequence of a fundamental observation of Atiyah and Bott [AB]. If we write them in terms of a real and complex moment map we obtain

$$\mu_c(A) = F_A \in \Omega^2(\Sigma; \mathrm{ad}P \otimes \mathbf{C})$$
$$\mu_r(A) = F' - F'' \in \Omega^2(\Sigma; \mathrm{ad}P)$$

where F' and F'' are the curvatures of the unique G-connections ∇' and ∇'' such that

$$(\nabla')^{1,0} = \nabla_A^{1,0} \qquad (\nabla'')^{0,1} = \nabla_A^{0,1}$$

Thus $\mu_c^{-1}(0)$ is the space of flat connections on the principal G^c-bundle.

A theorem of Donaldson [D2] and Corlette [Co] shows that each irreducible $\mathcal{G}^c$-orbit contains a solution to $\mu_r = 0$, which is unique modulo $\mathcal{G}$, so that the complex structure of the hyperkähler quotient can be identified with the space of flat irreducible connections modulo complex gauge equivalence. But the holonomy identifies this with the moduli space of representations $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\Sigma), G^c)/G^c$.

6.4. The proof of Donaldson and Corlette's theorem involves a reinterpretation of the moment map equations. The set-up for the quotient construction above involves a fixed metric on a principal bundle and an equivalence class of flat connections. Alternatively, we can consider a fixed flat connection and then solve for a metric satisfying $F' = F''$. A metric, compatible with the G-structure, is a section of the associated flat G^c/G-bundle and then the equations to be satisfied are equivalent to the statement that the section is *harmonic*. The non-positivity of the curvature of G^c/G then yields, via the Eells-Sampson theorem [ES], the existence result.

The analysis required to rigorously produce the moduli space with its metric involves Sobolev spaces for compact manifolds and is quite standard (see [H5]).

6.5. This hyperkähler manifold, of dimension $4(g-1)\dim G$, where g is the genus of Σ, shares a number of properties with the simple Eguchi-Hanson metric. The complex structure of the moduli space of representations constructed above is that of an affine variety, like the affine quadric. On the other hand there is one complex structure (and its conjugate) out of the 2-sphere generated by I, J and K which is not affine but which instead contains as a dense open set the cotangent bundle of the moduli space of stable G^c-bundles. This is the analogue of the cotangent bundle of $\mathbf{C}P^1$ for the Eguchi-Hanson metric. With this complex structure, the hyperkähler manifold can be identified as the moduli space of stable *Higgs bundles* (or stable pairs) on Σ [H5]. In this case the compactness of the surface imposes boundary conditions on the moment map equations which have as yet defied explicit solution.

7. RATIONAL MAPS

7.1. The quotient constructions offered as examples in Sections 5,6 and 7 are based on equations in 0,1 and 2 dimensions. There is a system of equations in $\mathbf{R}^3$ which again yields moduli spaces which are hyperkählerian. These are the *Bogomolny equations* for magnetic monopoles.

We consider here a trivial principal G-bundle P over $\mathbf{R}^3$ where G is a compact Lie group, and the space $\mathcal{A}$ of connections on P. We put

$$M = \mathcal{A} \times \Omega^0(\mathbf{R}^3; \mathrm{ad}P)$$

A point of M thus consists of a pair (A, ϕ) where A is a connection and ϕ a section of the adjoint bundle – the Higgs field.

The tangent space at a point is identified with the set of pairs

$$(\alpha, \phi) \in \Omega^1(\mathbf{R}^3; \mathrm{ad}P) \times \Omega^0(\mathbf{R}^3; \mathrm{ad}P)$$

and this has an obvious quaternionic structure by writing a tangent vector as

$$\phi + \alpha_1 i + \alpha_2 j + \alpha_3 k$$

There is an action of the group of gauge transformations $\mathcal{G}$ on M.

If we impose suitable boundary conditions (essentially comparison with a model as in Section 5) on (A, ϕ) and $\mathcal{G}$, then the L^2 inner product defines a hyperkähler metric on M and the action of $\mathcal{G}$ preserves it. We may then take a hyperkähler quotient.

7.2. If we write ∇_i for the directional covariant derivative defined by the connection A, the moment map equations become (cf 3.9)

$$[\phi, \nabla_1] + [\nabla_2, \nabla_3] = 0$$
$$[\phi, \nabla_2] + [\nabla_3, \nabla_1] = 0$$
$$[\phi, \nabla_3] + [\nabla_1, \nabla_2] = 0$$

or more compactly, using the Hodge star operator,

$$F_A = *\nabla_A \phi$$

which are the Bogomolny equations.

In the simplest case $G = SU(2)$, the boundary conditions on ϕ one usually takes imply that $\|\phi\| \to 1$ as $r \to \infty$. This yields an integer k, the degree of the map from a large sphere in $\mathbf{R}^3$ to the unit sphere in $\mathfrak{g} \cong \mathbf{R}^3$. This is called the *charge* of the solution.

7.3. The moduli space here can be identified by a supplementary theorem (due to Donaldson [D1]) with the space of based rational maps $f : \mathbf{C}P^1 \to \mathbf{C}P^1$ of degree k. The actual identification involves choosing a direction $\mathbf{u}$ and studying a scattering problem for the ordinary differential equation $(\nabla_{\mathbf{u}} + i\phi)s = 0$.

We may write such a rational map in the form

$$f(z) = \frac{a_0 + \ldots + a_{k-1}z^{k-1}}{b_0 + b_1 z + \ldots + z^k} = \frac{p(z)}{q(z)}$$

where the base-point ∞ is mapped to 0. Here, for f to be of degree k, $p(z)$ and $q(z)$ have no common factor. The space of such maps is clearly a complex manifold of real dimension $4k$. As a hyperkähler manifold it has a complex symplectic form. This is obtained as follows.

First factorize $q(z) = (z - \beta_1)\ldots(z - \beta_k)$. Since p and q have no common factor $p(\beta_i) \neq 0$. Then the form

$$\omega = \sum_{i=1}^{k} \frac{dp(\beta_i) \wedge d\beta_i}{p(\beta_i)}$$

extends to a holomorphic symplectic form on the space of rational maps. It is covariant constant with respect to the hyperkähler metric.

7.4. The rigorous construction of these metrics requires some analysis which was produced by C. H. Taubes. An account may be found in [AH] of the use of these results and properties of the metrics. There is one particular feature which distinguishes this family from the previous ones. The identification with the space of rational maps involves a choice of direction in $\mathbf{R}^3$. On the other hand the Bogomolny equations themselves are $SO(3)$-invariant. Thus $SO(3)$ acts transitively on the 2-sphere of complex structures on the moduli space which are therefore all equivalent. This is also true of the Taub-NUT metric, whose twistor space as described in (2.4) clearly inherits the action of $SU(2)$ on $\mathbf{C}P^1$.

These monopole metrics are known insofar as their twistor spaces can be described exactly (see [AH]). In the case of $k = 1$ and 2 they have also been computed explicitly– in the first case it is the flat metric on $S^1 \times \mathbf{R}^3$, and in the second a metric whose description involves elliptic integrals [AH].

8. LOOP GROUPS

8.1. If G is a compact Lie group, and $LG = \mathrm{Map}(S^1; G)$ denotes the space of smooth maps from the circle to G then the quotient LG/G by the subgroup G of constant maps is well-known to have the structure of an infinite-dimensional symplectic manifold [PS]. Indeed it also has a natural Kähler metric. The symplectic form is obtained by translation from the identity of the skew form

$$\omega(f,g) = \int_{S^1} \mathrm{Tr}(fg')d\theta$$

for $f, g : S^1 \to \mathfrak{g}$.

Clearly this definition holds when G is replaced by a complex semisimple group G^c, so that the complex manifold LG^c/G^c has a natural holomorphic symplectic form. Donaldson [D3] has shown how to give this space a natural hyperkähler metric.

8.2. The approach is to consider a trivial principal G-bundle over the unit disc $D \subset \mathbf{C}$ and the space $\mathcal{A}^c$ of connections on the associated G^c-bundle P^c which are smooth up to the boundary. The setting is therefore similar to that in Section 6, but now we have a non-trivial boundary, the circle S^1. However, if we consider the group of gauge transformations $\mathcal{G}$ which restrict to the identity on the boundary, then the hyperkähler moment maps are the same as in (6.2)

$$\mu_c(A) = F_A \in \Omega^2(D; \mathrm{ad}P \otimes \mathbf{C})$$
$$\mu_r(A) = F' - F'' \in \Omega^2(D; \mathrm{ad}P)$$

8.3. To solve the moment map equations, the harmonic map formulation is used again. In this case all flat connections are trivial, since the disc is simply-connected, so the problem concerns harmonic maps $f : D \to G^c/G$. Results of R. Hamilton [Ha] show that the Dirichlet problem can be solved in this case. Thus, given a map on the boundary circle, there is a unique harmonic map extending it to the disc.

In the formulation of connections, this means that given the boundary value of a flat G^c-connection on D, there is a unique solution of the hyperkähler moment map equations, modulo gauge transformations which preserve the boundary value. But since all such connections on the circle are gauge-equivalent to the trivial connection, which has automorphism group G^c, the hyperkähler quotient we are considering is

$$\mathrm{Map}(S^1; G^c)/G^c$$

8.4. The holomorphic form ω of Section 6 restricts to the space of flat connections $\mu_c^{-1}(0)$ to give

$$\omega(\alpha, \beta) = \int_D \mathrm{Tr}(\alpha \wedge \beta)$$

But, to be tangential at A to a flat connection, α and β must satisfy $\alpha = d_A\varphi$ and $\beta = d_A\psi$ for some $\varphi, \psi \in \Omega^0(D; \mathrm{ad}P^c)$ and hence

$$\begin{aligned}\omega(\alpha, \beta) &= \int_D \mathrm{Tr}(d_A\varphi \wedge d_A\psi) \\ &= \int_{S^1} \mathrm{Tr}(\varphi d_A\psi)\end{aligned}$$

which is clearly the skew form defined above on boundary values of maps $f : D \to G^c$.

8.5. Finally note that the moment map equations for this example can be put in the more familiar form of (3.9) by means of some substitutions. If we set

$$\nabla = (\nabla' + \nabla'')/2 \qquad \phi = (\nabla' - \nabla'')/2$$

where $\phi = \phi_1 dx_1 + \phi_2 dx_2$, then the equations may be written as

$$[\nabla_1, \nabla_2] + [\phi_1, \phi_2] = 0$$
$$[\nabla_1, \phi_2] + [\nabla_2, \phi_1] = 0$$
$$[\nabla_1, \phi_1] + [\phi_2, \nabla_2] = 0$$

9. CONCLUSIONS

9.1. The quotient construction yields a vast number of hyperkähler manifolds, in fact such a large number that a secondary task now presents itself to instil some order amongst them. This is particularly important because even those exemplified here have interrelationships – hyperkähler metrics may appear on the same space through different constructions.

Kronheimer's coadjoint orbits provide an example. The construction in Section 6 is given as an infinite-dimensional quotient based on ordinary differential equations. On the other hand D. Burns [Bu] has given an algebraic twistor construction (characteristically manifesting difficulty in showing completeness) for metrics on the same spaces. It is also true that Donaldson's complex loop group metric in Section 8 contains the same coadjoint orbits as fixed point sets of circle actions. Also, some of these spaces (as the Eguchi-Hanson metric shows) can be obtained as finite-dimensional quotients. Proving these to be isometric is not always easy.

Another class of examples are the monopole moduli spaces, which acquired a hyperkähler metric through an infinite-dimensional quotient based on the Bogomolny equations in 3 dimensions. There is however, through the

Nahm transform, a way of defining a hyperkähler metric through Nahm's equations in 1 dimension. Results such as those of Nakajima [N] have shown that these particular metrics coincide. A similar situation holds for instanton moduli spaces where results [BvB], [Ma] show the coincidence of metrics defined by two methods.

9.2. There are also quotient constructions for a class of manifolds related to the quaternions in a slightly more general manner than hyperkähler manifolds. We have already encountered *hypercomplex* manifolds, but there is also the class of *quaternionic Kähler* manifolds, which are Riemannian manifolds with holonomy group $Sp(n).Sp(1)$ like $\mathbf{H}P^n$ and *quaternionic* manifolds, which admit a torsion-free $GL(n,\mathbf{H}).Sp(1)$ connection [S]. For all of these [GL],[J] there are quotient constructions, but so far with a lesser range of examples than the hyperkähler quotient.

9.3. One final question concerns the compact examples like the K3 surface where we still rely entirely on existence theorems. Could they be obtained by the quotient construction? In particular, through a finite-dimensional quotient of a vector space?

The answer is no. Compact examples of finite-dimensional Kähler quotients, like the projective space in (3.5), may exist but the special curvature of a hyperkähler manifold prevents this from happening in the hyperkähler context. Perhaps the easiest way to see this is in the 4-dimensional case.

Suppose that a compact hyperkähler manifold M^4 is obtained by a quotient of a linear hyperkähler action of G on $\mathbf{H}^n$. Then if μ is the hyperkähler moment map, $\mu^{-1}(\zeta)$ is a principal G-bundle P over M. The flat metric on $\mathbf{H}^n$ induces a G-invariant metric on P and the orthogonal complements to the orbit directions define a connection on P. Now for each complex structure on $\mathbf{H}^n$ (and hence M), the associated principal G^c-bundle is $\mu_c^{-1}(\alpha)$ which is holomorphic. This implies that the connection is compatible with all three complex structures I, J and K and hence (see [AHS]) is a solution to the anti-self dual Yang-Mills equations on M. According to the Atiyah-Ward construction, this corresponds to a holomorphic principal bundle on the twistor space Z of M which is, not surprisingly, $\tilde{\mu}^{-1}(\zeta)$.

The connection has a further property, for we have an embedding of $\tilde{\mu}^{-1}(\zeta)$ in the twistor space $\mathbf{C}^{2n}(1)$ corresponding to the embedding $P = \mu^{-1}(\zeta) \subset \mathbf{H}^n$. This embedding defines an equivariant section of $\mathbf{C}^{2n}(1)$ on the principal G^c-bundle over Z and this (see [H2]) has an interpretation via the Penrose transform as the solution ψ of a differential equation on M – the *twistor equation* coupled to the anti-self-dual Yang-Mills connection. Now for such an object there is a Weitzenböck formula which gives a vanishing theorem on a *compact* manifold. In our case, since the Ricci tensor of a hyperkähler manifold vanishes, then in particular so does the scalar curvature, and the corresponding vanishing theorem [H2] implies covariant constancy of ψ. This in turn means that the map $P \subset \mathbf{H}^n$ is constant in horizontal directions which is a contradiction to it being an embedding.

9.4. As for infinite-dimensional quotients, there remains a hope. Instanton moduli spaces on hyperkähler manifolds have induced hyperkähler metrics through the moment map interpretation of the anti-self-dual Yang-Mills equations, but they tend to be either non-compact or have singularities corresponding to reducible connections. However, introducing singularities into the connections themselves may yet induce more regularity into the moduli space. Ultimately, it is possible that the K3 surface may be brought into the fold of hyperkähler quotients. It does not necessarily mean that we can write down the metric (a glance at [AH] will show that explicitness is not gained easily) but that we shall have in any case a deeper understanding of the pervasiveness of the geometry which has arisen from the quaternions.

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