# WHAT IS...

# a Higgs Bundle?

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A Higgs bundle is a holomorphic vector bundle together with a Higgs field. Such objects first emerged twenty years ago in Nigel Hitchin's study of the self-duality equations on a Riemann surface and in Carlos Simpson's Ph.D. thesis and subsequent work on nonabelian Hodge theory. Hitchin introduced the term "Higgs field" because of similarities to objects labeled this way in other equations of gauge theory. In those contexts Higgs fields describe physical particles like the Higgs boson. Simpson suggested the shorthand "Higgs bundle" for a bundle together with a Higgs field.

Higgs bundles have a rich structure and play a role in many different areas including gauge theory, Kähler and hyperkähler geometry, surface group representations, integrable systems, nonabelian Hodge theory, the Deligne–Simpson problem on products of matrices, and (most recently) mirror symmetry and Langlands duality. In this essay we will touch lightly on a selection of these topics.

We start with the definition: *A Higgs bundle is a pair* $(E, \phi)$ *where* $E$ *is a holomorphic vector bundle and* $\phi$*, the Higgs field, is a holomorphic* 1*-form with values in the bundle of endomorphisms of* $E$*, satisfying* $\phi \wedge \phi = 0$.

In the simplest examples the bundle is a complex line bundle and the Higgs field is a holomorphic 1-form. To see a nonabelian example, set $E = K^{1/2} \oplus K^{-1/2}$ where $K^{1/2}$ is a complex line bundle whose square is $K$, the canonical bundle on a Riemann surface (i.e., the bundle of holomorphic 1-forms). A Higgs field on $E$ is then equivalent to a bundle map $\phi : E \longrightarrow E \otimes K$. We obtain a family of Higgs fields on $E$ parameterized by quadratic differentials, i.e., sections $a$ of the line bundle $K^2 \cong \mathrm{Hom}(K^{-1/2}, K^{1/2} \otimes K)$, by setting $\phi = \left(\begin{smallmatrix} 0 & a \\ 1 & 0 \end{smallmatrix}\right)$, where 1 is the identity section of the trivial bundle $\mathrm{Hom}(K^{1/2}, K^{-1/2} \otimes K)$.

We now look at how Higgs bundles emerge in nonabelian Hodge theory. Hodge theory uses harmonic differential forms to represent de Rham cohomology classes on Riemannian manifolds. On a hermitian manifold, say $X$, $\bar{\partial}$-harmonic forms give analogous representatives for Dolbeault cohomology classes. If the metric on $X$ is Kähler, the real and complex theories are compatible. This relates topological and holomorphic data on $X$ and reveals additional structure on the topological side, i.e., on the cohomology groups $H^k(X; \mathbb{C})$. For $k = 1$ we get

$$H^1(X; \mathbb{C}) \cong H^{0,1}(X) \oplus H^{1,0}(X). \tag{1}$$

On the holomorphic side, $H^{0,1}(X)$ describes deformations of holomorphic line bundles on $X$. The holomorphic data thus come from a pair $(E, \phi)$, where $E$ is a line bundle and $\phi \in H^{1,0}(X)$ is a holomorphic 1-form, i.e., it comes from an abelian Higgs bundle. On the topological side, $H^1(X; \mathbb{C})$ models the tangent space to the space of homomorphisms from $\pi_1(X)$ to $\mathbb{C}^*$. This is the same as the space of flat complex line bundles on $X$.

In the nonabelian theory we replace $\mathbb{C}^*$ by a nonabelian Lie group. For definiteness we take $\mathrm{SL}(n, \mathbb{C})$. The topological side of the Hodge theory now has $\mathrm{SL}(n, \mathbb{C})$-representations of $\pi_1(X)$ or, equivalently, flat complex vector bundles on $X$. A theorem of Corlette and Donaldson provides the harmonic part of the theory; it says that if the representation of $\pi_1(X)$ is completely reducible, then the corresponding flat bundle, say $E$, supports a *harmonic metric* (solving an appropriate generalization of Laplace's equation). The holomorphic interpretation uses the fact that flat structures are defined by bundle connections with vanishing curvature. The harmonic metric splits the flat connection into two parts: a skew-hermitian (unitary) part and a hermitian

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part. The anti-holomorphic component of the former defines a holomorphic structure on $E$; the latter defines a holomorphic endomorphism-valued 1-form, i.e., a Higgs field $\phi$. The holomorphic data, i.e. $(E, \phi)$, thus define a Higgs bundle.

We next explore some features of Higgs bundles, starting with a theorem of Hitchin and Simpson that says that, for a Higgs bundle to admit a harmonic metric as above, it must satisfy a condition called *stability*. Together with Corlette's theorem this establishes a correspondence between stable Higgs bundles on a Kähler manifold and irreducible $\mathrm{SL}(n, \mathbb{C})$-representations of $\pi_1(X)$. This is a Higgs bundle version of a famous theorem of Narasimhan–Seshadri on vector bundles and its generalization by Donaldson and Uhlenbeck–Yau.

A key attribute of Higgs bundles is a $\mathbb{C}^*$-action given by $\lambda(E, \ \phi) := (E, \ \lambda\phi)$. Isomorphism classes of Higgs bundles fixed by this action are complex variations of Hodge structure (the focus of Simpson's Ph.D. thesis). Through them Higgs bundles reveal strong restrictions on fundamental groups of compact Kähler manifolds. Using the fact that stable Higgs bundles form a Tannakian category, this $\mathbb{C}^*$-action also reveals a $\mathbb{C}^*$-action on the pro-reductive completion of $\pi_1(X)$.

Another central feature of Higgs bundles is that they have continuous moduli, i.e., they come in families parameterized by the points of a geometric space (in fact a quasi-projective variety) known as a moduli space. One method for constructing such spaces, using Mumford's geometric invariant theory (GIT), depends on a property called stability. When $X$ is a Riemann surface (assumed from now on) the previously mentioned stability property corresponds precisely to the GIT notion. The essence of nonabelian Hodge theory thus amounts to an identification between the moduli space of stable Higgs bundles on the Riemann surface $X$ and the moduli space of irreducible $\mathrm{SL}(n, \mathbb{C})$-representations of its fundamental group. In the abelian counterpart all Higgs bundles are stable and the space of holomorphic 1-forms is dual to the infinitesimal deformation space of a line bundle. Thus the moduli space is the cotangent bundle to the Jacobian variety of $X$. The corresponding representation space is now the character variety $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \cong (\mathbb{C}^*)^{2g}$.

The moduli space has a third description as a space of solutions to the *Hitchin equations*. These are gauge-theoretic equations for the Higgs field, $\phi$, and an $\mathrm{SU}(n)$ connection $A$ compatible with the holomorphic structure on the bundle $E$:

$$\begin{aligned} F_A + [\phi, \phi^*] &= 0 \\ d''_A \phi &= 0. \end{aligned}$$

Here $F_A$ is the curvature of $A$, and $d''_A \phi$ is the anti-holomorphic part of the covariant derivative of $\phi$. Hitchin obtained these equations by considering instantons (solutions to the anti-self-duality equations) that are invariant under a two-dimensional group of symmetries on a four-dimensional manifold. The equations express both the flatness of an $\mathrm{SL}(n, \mathbb{C})$-connection $A + \phi + \phi^*$ and the harmonicity condition for a metric in the resulting flat bundle. This links flat bundles and Higgs bundles in the correspondence described earlier.

The four-dimensional origin and basic structure of the equations account for a hyperkähler structure on the moduli space. This is a Riemannian metric that is Kähler with respect to three distinct complex structures defined by operators $I$, $J$, and $K$ satisfying the quaternionic relations. The moduli spaces of Higgs bundles on Riemann surfaces are noncompact hyperkähler manifolds. The restriction of the $\mathbb{C}^*$-action to $S^1$ is Hamiltonian with respect to one of the Kähler forms on the moduli space. The associated symplectic moment map is given by the $L^2$-norm of the Higgs field. This map constitutes a perfect Bott–Morse function on the moduli space and provides a powerful tool for studying its topology.

In addition to providing a distinguished Morse function, the Higgs field is responsible for another signature feature of the Higgs bundle moduli space $M$, namely the Hitchin fibration. Since $\phi$ takes its values in endomorphisms of the bundle fibers, we can compute $\det(\phi - \lambda I)$. The coefficients of this characteristic polynomial define the *Hitchin map*

$$H : M \longrightarrow \bigoplus_{d=2}^{n} H^0(X; K^d).$$

Here $K$ is the canonical bundle on $X$. The target is a vector space with dimension half that of $M$ and the generic fiber is an abelian variety, in fact the Jacobian of the so-called spectral curve. This is an example of an algebraically completely integrable system.

The Hitchin map has a section whose image is a component of the moduli space of representations of $\pi_1(X)$ in $\mathrm{SL}(n, \mathbb{R})$. It is a $(n^2 - 1)(g - 1)$-dimensional complex cell that for $n = 2$ corresponds to the Teichmüller space of the surface. The Higgs bundles are precisely the rank 2 examples described earlier. Applying the existence theorem for Hitchin's equations to those with $a = 0$ provides a new proof of the uniformization theorem for Riemann surfaces.

Virtually everything described above applies if $\mathrm{SL}(n, \mathbb{C})$ is replaced by a complex semisimple Lie group $G$. The resulting $G$-Higgs bundle theory introduces holomorphic tools for studying representations of $\pi_1(X)$ into $G$ and also into the real forms of $G$. In a sign that they still have much to teach us, Higgs bundles play a role in the recent Kapustin–Witten interpretation via topological field theory of the geometric Langlands correspondence. We end by noting that the Hitchin equations emerge here after imposing a two-dimensional symmetry—much as in Hitchin's original derivation of his equations!

## References


[1] For the original pioneering work see N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc.* (3) **55** (1987) 59–126; and C. T. Simpson, Higgs bundles and local systems, *Publ. Math. I. H. E. S.* **75** (1992) 5–95.

[2] For references to subsequent developments see S. Bradlow, O. García-Prada, and P. B. Gothen, Surface group representations and $\mathrm{U}(p, q)$-Higgs bundles, *J. Diff. Geom.* **64** (2003) 111–170; and R. Donagi and T. Pantev, Langlands duality for Hitchin systems, `arxiv:math/0604617` (2006).