Locality and Quantum Physics

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Introduction

There is always a tension between the fact that we have access only to small systems (i.e. with a small number of relevant degrees of freedom) and the fact that these small systems are (not completely isolated) parts of a bigger system.

In classical physics this is not that dramatic. Once we know e.g. the electromagnetic field F locally we also know it globally. This may be formalized by choosing an open cover (U_i) of spacetime, selecting 2-forms F_i on U_i satisfying Maxwell's equations

$$dF_i = 0 \ , \delta F_i = 0$$

and coinciding on intersections,

$$F_i = F_j \text{ on } U_i \cap U_j \ ,$$

then there is a unique solution F on the whole spacetime, given by

$$F = \sum F_i \chi_i$$

with some partition of unity adapted to the covering.

In quantum physics, the situation is different: Consider a system consisting of 2 independent spin-$\frac{1}{2}$ systems. The observables of the whole system form the 4×4-matrices where the subsystems are given by matrices of the form $A \otimes 1_2$ and $1_2 \otimes B$, respectively, with 2×2-matrices A, B and the unit matrix 1_2 in 2 dimensions.
A typical (pure) state is given by the density matrix

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

The induced states on the subsystems are then obtained by a partial trace and given by the density matrices

$$\rho_1 = \rho_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \, .$$

The partial states will not change if we transform ρ by a unitary of the form $U = U_1 \otimes U_2$,

$$\rho \mapsto U^* \rho U \ ,$$

with unitary 2×2-matrices U_1, U_2,

$$\mathrm{tr} U^* \rho U (A \otimes 1_2) = \mathrm{tr} \rho_1 U_1 A U_1^* = \mathrm{tr} \rho_1 A \ .$$

Hence it is impossible to retrieve the state of the full system from the states of the subsystems. Using Bell's inequalities, one can even show that ρ cannot be approximated by mixtures of states of product form (separable states),

$$\rho' = \sum \lambda_i \rho_1^{(i)} \otimes \rho_2^{(i)}, \ \lambda_i \geq 0, \ \sum \lambda_i = 1$$

One says, that ρ is entangled.

States of the form above have to be carefully prepared and one might object that the correlations have been imposed by the preparation.
A surprising effect without any joint preparation is the Hanbury Brown-Twiss effect.

There astronomers measure correlations of light intensities from far away sources. If we understand light emission as the creation of a photon, the effect concerns coincidences of 2 completely independent photons which arrive at two spacelike separated detectors.

The observed correlations are due the fact that the photons are indistinguishable and satisfy Bose statistics. Therefore the 2-photon states cannot have product form and are necessarily entangled.

Observables and states

In quantum physics it has turned out to be useful to distinguish observables and states. Observables may be considered as measuring procedures, whereas states are procedures for preparing a system for measurement. It is a matter of choice whether part of the preparation is subsumed into the measurement procedure or not.

Mathematically, the observables can be considered as elements of an associative unital $\star$-algebra (preferably a C*- or von Neumann algebra in order to have nice functional analytic structures). States associate to every selfadjoint element a probability measure for the outcome of a measurement. It is convenient to realize them as linear functionals ω on the algebra which satisfy the conditions of positivity

$$\omega(A^*A) \geq 0$$

and are normalized

$$\omega(1) = 1 \ .$$

The values of the functional are interpreted as expectation values, and the full probability distribution can be obtained from its moments given by $\omega(A^n)$.

Choices for observables and states:

Quantum mechanics:

Observables are bounded operators on some Hilbert space and pure states are unit rays of the Hilbert space. Mixed states are density matrices with the rank 1 density matrices ($1d$-projections) as pure states.

Advantage: Hilbert space techniques allow very detailed calculations and estimates for systems with small number of particles.

Disadvantage: For large systems one often has to consider limits in form of convergence of expectation values where the limit cannot be described by a density matrix in the original Hilbert space.

Wightman functional:
The theory is defined in terms of a sequence W_n of distributions in n variables which has to satisfy certain axioms (the Wightman axioms).

These distributions are interpreted as the expectation values of products of fields at different points of spacetime. In this case the algebra of observables is the Borchers-Uhlmann algebra i.e. the tensor algebra over the test function space with complex conjugation as involution.

The sequence $(W_n)_{n\in\mathbb{N}_0}$ yields a state W on this algebra by

$$W(f_1 \otimes \cdots \otimes f_n) := W_n(f_1 \otimes \cdots \otimes f_n) \ .$$

A variant of this is the euclidean path integral where the correlation functions, analytically continued to the euclidean region, are the correlation functions of a probability distribution.

If the latter satisfy the Osterwalder-Schrader axioms, the Wightman functions can be rediscovered, again by analytic continuation.

In these formulations all dynamical and algebraic information is contained in the state whereas the algebra only depends on the type of fields which occur. Therefore, nonlocal and local features are mixed.

In particular it is not obvious how different states of the same system are characterized. This problem occurs e.g. for states with different temperatures.

Algebraic formulation

For a suitable family $\mathfrak{K}$ of globally hyperbolic subregions $\mathcal{O}$ of some spacetime M one associates algebras of observables $\mathfrak{A}(\mathcal{O})$ satisfying the Haag-Kastler axioms:

Isotony: For each inclusion $\mathcal{O}_1 \subset \mathcal{O}_1$ there exists an embedding $i_{\mathcal{O}_2\mathcal{O}_1} : \mathfrak{A}(\mathcal{O}_1) \to \mathfrak{A}(\mathcal{O}_2)$ such that $i_{\mathcal{O}_3\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_3\mathcal{O}_1}$ holds whenever $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_3$.

Locality: If $\mathcal{O}_1 \cup \mathcal{O}_2 \subset \mathcal{O}$ and $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$, then

$$[i_{\mathcal{O}\mathcal{O}_1}(A), i_{\mathcal{O}\mathcal{O}_2}(B)] = 0$$

for all $A \in \mathfrak{A}(\mathcal{O}_1), B \in \mathfrak{A}(O_2)$.

Covariance: If the symmetries g of the spacetime induce bijections on $\mathfrak{K}$ then there exist isomorphisms $\alpha_g : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(g\mathcal{O})$ such that $\alpha_g \circ \alpha_h = \alpha_{gh}$.

Timeslice: If $\mathcal{O} \subset \mathcal{O}_1$ contains a Cauchy surface of $\mathcal{O}_1$ then $i_{\mathcal{O}_1\mathcal{O}}$ is an isomorphism.

If $\mathfrak{K}$ as a partially ordered set with respect to inclusion is directed, i.e. for all $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{K}$ there exists some $\mathcal{O} \in \mathfrak{K}$ such that

$$\mathcal{O}_1 \cup \mathcal{O}_2 \subset \mathcal{O} \ ,$$

then one can associate a unique C*-algebra $\mathfrak{A}(M)$ to the spacetime M which is characterized by the following conditions:

- There exist homomorphisms $i_{\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(M)$ such that

  $$i_{\mathcal{O}} \circ i_{\mathcal{O}\mathcal{O}_1} = i_{\mathcal{O}_1}$$

  whenever $\mathcal{O}_1 \subset \mathcal{O}$.
- For any C*-algebra $\mathfrak{B}$ and homomorphisms $j_{\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{B}$ satisfying

  $$j_{\mathcal{O}} \circ i_{\mathcal{O}\mathcal{O}_1} = j_{\mathcal{O}_1}$$

  there is a unique homomorphism $\varphi : \mathfrak{A}(M) \to \mathfrak{B}$ such that

  $$j_{\mathcal{O}} = \varphi \circ i_{\mathcal{O}} \ .$$

$\mathfrak{A}(M)$ is called the inductive limit of the Haag-Kastler net on M and is also known as the algebra of quasilocal observables. It has the following nice properties

- The homomorphisms $i_{\mathcal{O}}$ are faithful and hence isometric.
- $\bigcup_{\mathcal{O}} i_{\mathcal{O}}(\mathfrak{A}(\mathcal{O}))$ is dense in $\mathfrak{A}(M)$.
- The closed ideals in $\mathfrak{A}(M)$ are generated by local elements $i_{\mathcal{O}}(A)$. In case the algebras $\mathfrak{A}(O)$ are simple also $\mathfrak{A}(M)$ is simple.
- Spacetime symmetries act by automorphisms on $\mathfrak{A}(M)$.

Topological obstructions

Complications arise if the set of subregions $\mathfrak{K}$ is not directed.

Let us look at the example of chiral conformal fields which can be described by a Haag-Kastler family of algebras associated to nondense and nonempty open intervals $\mathcal{I} \subset \mathbb{S}^1$. Since the set of these intervals is not directed, the inductive limit construction no longer works.

Nevertheless, the universality conditions for the algebra associated to the full space $\mathbb{S}^1$ remain meaningful and characterize a unique (up to isomorphy) C*-algebra which in terms of category theory is a so-called colimit.

This colimit, however, has in general a more complicated structure than the inductive limit. First of all, it might happen that the homomorphisms $i_{\mathcal{I}} : \mathfrak{A}(\mathcal{I}) \to \mathfrak{A}(\mathbb{S}^1)$ are not injective.

Typically, however, one starts from a family of faithful representations $\pi_{\mathcal{I}}$ of $\mathfrak{A}(\mathcal{I})$ on a fixed Hilbert space $\mathcal{H}$ such that

$$\pi_{\mathcal{I}} \circ i_{\mathcal{I}\mathcal{I}_1} = \pi_{\mathcal{I}_1}$$

for $\mathcal{I}_1 \subset \mathcal{I}$. Then, by the universality condition, there is a unique representation π of $\mathfrak{A}(\mathbb{S}^1)$ on $\mathcal{H}$ with

$$\pi \circ i_{\mathcal{I}} = \pi_{\mathcal{I}} \ ,$$

and therefore also $i_{\mathcal{I}}$ has to be faithful.

But the global algebra $\mathfrak{A}(\mathbb{S}^1)$ may contain nontrivial ideals even if all the local algebras $\mathfrak{A}(\mathcal{I})$ are simple. These ideals are, in general, related to topological invariants of the spacetime.

As an example we consider a chiral subalgebra of the free massless field φ in 2 dimensions. It is generated by the chiral currents $j = \partial_u \varphi du$ (in lightcone coordinates). A C*-algebra $\mathfrak{A}(\mathcal{I})$ associated to an interval $\mathcal{I}$ is generated by the Weyl operators

$$W(f) = e^{i \int f j}$$

with smooth real valued functions f with supp$f \subset \mathcal{I}$ and the product

$$W(f)W(g) = e^{-\frac{i}{2} \int f dg} W(f+g) \; .$$

Since the antisymmetric form $\sigma(f,g) = \int f dg$ is non degenerate on $\mathcal{D}(\mathcal{I})$, the algebras $\mathfrak{A}(\mathcal{I})$ are simple. The algebra associated to $\mathbb{S}^1$, however, contains also the Weyl operators $W(c)$ with the constant function c on $\mathbb{S}^1$. But the extension of σ to all smooth functions on $\mathbb{S}^1$ is degenerate, and the algebra has a nontrivial center generated by the operators $W(c)$, and hence also nontrivial ideals.

Another example is the even subalgebra for chiral Majorana fermions on $\mathbb{S}^1$. The algebra of canonical anticommutation relations $\mathrm{CAR}(H)$ over a real Hilbert space H is the unital C*-algebra characterized by the condition:

- There is a real linear map $B : H \to \mathrm{CAR}(H)$ such that $B(f)^* = B(f)$ and
$$B(f)^2 = ||f||^2 \ .$$
- Given any unital C*-algebra $\mathfrak{B}$ and any real linear map $B' : H \to \mathfrak{B}$ satisfying the above conditions then there is a unique homomorphism $\varphi : \mathrm{CAR}(H) \to \mathfrak{B}$ with $\varphi \circ B = B'$.

The even subalgebra is the unital C*-algebra $\mathrm{CAR}(H)_{even}$ generated by the elements $b(f,g) = B(f)B(g)$. It is characterized by the relations:

$$
\begin{aligned}
&b : H \times H \to \mathrm{CAR}(H)_{even} \text{ is bilinear} \\
&b(f,f) = ||f||^2 \\
&b(f.g)b(g,h) = ||g||^2 b(f,h) \\
&b(f,g)^* = b(g,h)\ .
\end{aligned}
$$

Let $H_\mathcal{I} = L^2(\mathcal{I},\mathbb{R})$ with natural embeddings $H_\mathcal{I} \hookrightarrow H_\mathcal{J}$ for $\mathcal{I} \subset \mathcal{J}$. We then have the algebras $\mathfrak{A}(\mathcal{I}) = \mathrm{CAR}(H_\mathcal{I})_{even}$ with the induced homomorphisms $i_{\mathcal{J}\mathcal{I}}$. The algebra $\mathfrak{A}(\mathbb{S}^1)$ is generated by the elements

$$b_\mathcal{I}(f,g) = i_\mathcal{I}(b(f,g)),$$

$f,g \in H_\mathcal{I}$ with the obvious relations. There is, however, no a priori relation between $b_\mathcal{I}(f,g)$ and $b_\mathcal{J}(f,g)$ if $\mathcal{I} \cup \mathcal{J} = \mathbb{S}^1$ and $\mathrm{supp} f, \mathrm{supp} g \subset \mathcal{I} \cap \mathcal{J}$.

Let $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$ and $\mathcal{J}_\pm \supset \mathcal{I}_1 \cup \mathcal{I}_2$, $\mathcal{J}_+ \cup \mathcal{J}_- = \mathbb{S}^1$, and let $f \in H_{I_1}$. $g \in H_{\mathcal{I}_2}$ with $||f|| = ||g|| = 1$. Then

$$Y = b_{\mathcal{J}_+}(f,g) b_{\mathcal{J}_-}(g,f)$$

has the following properties:

Y is independent of the choices of $f, g, \mathcal{I}_1, \mathcal{I}_2, \mathcal{J}_\pm$

$Y \in \mathrm{Z}(\mathfrak{A}(\mathbb{S}^1))$

$Y^2 = 1$

$Y^* = Y$

The algebra is therefore a direct sum

$$\mathfrak{A}(\mathbb{S}^1) = \mathfrak{A}(\mathbb{S}^1)_+ \oplus \mathfrak{A}(S^1)_-$$

corresponding to the eigenvalues of Y. This decomposition refers to the Ramond (periodic b.c.) and Neveu-Schwarz (anti-periodic b.c.) sectors.

Generalizations

The construction of the colimit corresponds to families of representations $\pi_{\mathcal{O}}$ of $\mathfrak{A}(\mathcal{O})$ in some Hilbert space $\mathcal{H}$ with the compatibility condition

$$\pi_{\mathcal{O}} = \pi_{\mathcal{O}_1} \circ i_{\mathcal{O}\mathcal{O}_1} \ .$$

Such families are in 1-1 correspondence to representations π of the colimit $\mathfrak{A}(M)$ with $\pi_{\mathcal{O}} = \pi \circ i_{\mathcal{O}}$.

A more general situation occurs in the presence of inner symmetries. An inner symmetry is a family $\alpha = (\alpha_{\mathcal{O}})_{\mathcal{O}}$ of automorphisms of $\mathfrak{A}(\mathcal{O})$ with

$$\alpha \circ i_{\mathcal{O}\mathcal{O}_1} = i_{\mathcal{O}\mathcal{O}_1} \circ \alpha$$

for $\mathcal{O}_1 \subset \mathcal{O}$.

One may then modify the embeddings $i_{\bullet\bullet}$ by choosing for each inclusion $\mathcal{O}_1 \subset \mathcal{O}_2$ an inner symmetry $\alpha^{\mathcal{O}_2\mathcal{O}_1}$ such that

$$\alpha^{\mathcal{O}_3\mathcal{O}_2}\alpha^{\mathcal{O}_2\mathcal{O}_1} = \alpha^{\mathcal{O}_3\mathcal{O}_1}$$

Then the transformed embeddings

$$i^{\alpha}_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_2\mathcal{O}_1} \circ \alpha^{\mathcal{O}_2\mathcal{O}_1}$$

satisfy the compatibility condition, i.e. for $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_3$

$$i^{\alpha}_{\mathcal{O}_3\mathcal{O}_2} \circ i^{\alpha}_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_3\mathcal{O}_2} \circ \alpha^{\mathcal{O}_3\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} \circ \alpha^{\mathcal{O}_2\mathcal{O}_1}$$

$$= i_{\mathcal{O}_3\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} \circ \alpha^{\mathcal{O}_3\mathcal{O}_1} = i^{\alpha}_{\mathcal{O}_3\mathcal{O}_1}$$

In case $\alpha^{\mathcal{O}_2\mathcal{O}_1} = \beta^{\mathcal{O}_2}(\beta^{\mathcal{O}_1})^{-1}$ for a family of inner symmetries $\beta^{\mathcal{O}}$, the new system is equivalent to the old one. This is always true if $\mathfrak{K}$ is simply connected.

In case $\mathfrak{K}$ is not simply connected one finds non isomorphic systems, e.g. Isham's twisted fields (see B. Lang, Ph.D. thesis, York 2014).

If the group of inner symmetries is compact one can pass to the subnet of invariants. There the embeddings are unique, and one obtains the colimit as before. The occurence of twists in the original net should then be visible in the structure of the center of the algebra, as observed in examples, but this has to be checked in the general case.

Conclusions and Outlook

Given the local algebras of observables one can construct the global algebra as a colimit. Its structure contains interesting topological information. In the presence of a compact group of inner symmetries the colimit of the net of algebras of invariant observables seems to be the appropriate object.

In the case of non compact symmetry groups a more general concept, as e.g. the homotopical colimit (Benini, Schenkel, Szabo), seems to be a promising choice.

An interesting and largely open problem is the generalization of the theory of superselection sectors to the generally locally covariant case. One expects new sectors corresponding to topological invariants. Some work in this direction was done by Brunetti and Ruzzi (2008), partially based on ideas of Roberts.