A homotopical algebra approach to gauge theories

Based on joint work with A. Schenkel and R.J. Szabo.

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Outline

1. Comparison between field theory without gauge and with gauge
2. Brief excursus on model categories
3. Groupoids of gauge theory on contractible manifolds
4. Restricting to the Abelian case
5. A mathematical “experiment”
6. Break!

Field theory (without gauge) 1

The relevant data:

- A spacetime manifold M (e.g. globally hyperbolic);
- A bundle E over M (typically a vector bundle);
- Field configurations, i.e. global sections of E;
- A partial differential equation (e.g. linear and hyperbolic) specifying the "time-evolution" of field configurations.

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A basic example, the scalar field:

- Consider the trivial vector bundle $M \times \mathbb{R} \to M$;
- A section is a smooth function $\phi : M \to \mathbb{R}$;
- Dynamics specified by the d'Alembert operator: $\Box\phi + m^2\phi = 0$.

Field theory (without gauge) 2

Observation:

- Field configurations form a set (possibly with further structure).
- There is a strict notion of equality between points of a set.

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In the language of category theory, a set is a category whose objects are the points and whose morphisms are only identities.

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Back to the scalar field:

- $C^\infty(M;\mathbb{R})$ is the set of field configurations (before dynamics).
- One can decide whether the equality $\phi = \phi'$ holds or not.

Field theory (without gauge) 3

- You are given a manifold M with an open cover $\{U_\alpha\}$;
- On each U_α, a scalar field $\phi_\alpha \in C^\infty(U_\alpha; \mathbb{R})$ is provided;
- $\phi_\alpha = \phi_\beta$ on $U_{\alpha\beta} = U_\alpha \cap U_\beta$ holds.

Then you can **glue** the data $\{\phi_\alpha\}$ to form a global $\phi \in C^\infty(M; \mathbb{R})$:

$$\phi = \phi_\alpha \text{ on each } U_\alpha.$$

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We can abstract from our notion of field theory (w/o gauge):

A **sheaf** is a functor $\mathfrak{C}$ from $\mathrm{Man}^{\mathrm{op}}$ to Sets fulfilling the following condition for each manifold M and each cover $\{U_\alpha\}$ of M:

$$\mathfrak{C}(M) \xrightarrow{\simeq} \lim \Big(\prod_\alpha \mathfrak{C}(U_\alpha) \rightrightarrows \prod_{\alpha\beta} \mathfrak{C}(U_{\alpha\beta}) \Big).$$

Observables for field theory (without gauge)

Observables are (a suitable algebra of) functionals on configurations.
Configurations form a sheaf, dually observables should form a cosheaf:

Global observables are equivalent to (co)gluing local ones:
$\mathfrak{O} : \mathsf{Man} \to \mathsf{Alg}$ is a functor such that

$$\mathfrak{O}(M) \xleftarrow{\simeq} \operatorname{colim}\Big(\coprod_{\alpha} \mathfrak{O}(U_\alpha) \leftleftarrows \coprod_{\alpha\beta} \mathfrak{O}(U_{\alpha\beta}) \Big).$$

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This idea is not new:

Fredenhagen's universal algebra (1982): The global algebra of observables $\mathfrak{O}(M)$ can be reconstructed from the local ones $\mathfrak{O}(U_\alpha)$.

However, for non-gauge theories, e.g. the scalar field, we are usually able to understand global configurations and observables directly.

Gauge field theory 1

The relevant data:

- A spacetime manifold M (e.g. globally hyperbolic);
- A principal G-bundle P over M with connection A;
- Field configurations are pairs (P, A);
- Gauge transformations given by $g : (P, A) \xrightarrow{\simeq} (P', A')$;
- A partial differential equation specifying the dynamics (e.g. hyperbolic after a suitable gauge fixing).

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A basic example, Yang-Mills on a contractible manifold M:

- Consider the principal bundle $M \times G \to M$ with connection A;
- A is specified by a one-form on M with values in $\mathfrak{g}$;
- A gauge transformation is specified by a smooth function $g : M \to G$ such that $A' = g\,A\,g^{-1} + g\,\mathrm{d}\,g^{-1}$;
- Dynamics via curvature: $F = \mathrm{d}\,A + A \wedge A$, $\mathrm{d} *F + A \wedge *F = 0$.

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Observation:

- Field configurations form a category (in fact, a groupoid).
- There is no notion of equality for objects in a category, but we have equivalences given by isomorphisms.

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Back to Yang-Mills:

- $G - \mathrm{Bun}_{\mathrm{conn}}(M)$ is the groupoid of bundle-connection pairs (P, A) together with gauge transformations.
- $(P, A) = (P', A')$ does not even make any sense, however one can still decide whether $(P, A) \simeq (P', A')$ or not by exhibiting an iso.

Gauge field theory 3

- You are given a manifold M with an open cover $\{U_\alpha\}$;
- On each U_α, a bundle-connection pair (P_α, A_α) is provided;
- On each overlap $U_{\alpha\beta}$ a gauge transformation $g_{\alpha\beta} : (P_\alpha, A_\alpha) \xrightarrow{\simeq} (P_\beta, A_\beta)$ is provided;
- Consistency condition on each triple overlap $U_{\alpha\beta\gamma}$: $g_{\beta\gamma}\, g_{\alpha\beta} = g_{\alpha\gamma}$.

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Then you can **glue** all (P_α, A_α) to form a global (P, A):

$$P = \big(\coprod_\alpha P_\alpha\big)/\sim_{g_{\alpha\beta}}, \quad A = A_\alpha \text{ on } U_\alpha.$$

Of course, the pair (P, A) we obtain is specified only up to (a unique) isomorphism!

Gauge field theory 4

With this in mind, we abstract from our notion of gauge theory:

A **stack** is a (pseudo)functor $\mathfrak{C}$ from $\mathrm{Man}^{\mathrm{op}}$ to Gpds (groupoids) fulfilling **descent** for each manifold M and each cover $\{U_\alpha\}$ of M. Roughly speaking, one can always glue coherent local data to global data up to a unique isomorphism.

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By [Hollander, Israeli J. Math. 163 (2008) 63], this is equivalent to:

A **homotopy sheaf** is a functor $\mathfrak{C}$ from $\mathrm{Man}^{\mathrm{op}}$ to Gpds fulfilling the following condition for each manifold M and each cover $\{U_\alpha\}$ of M:

$$\mathfrak{C}(M) \xrightarrow{\simeq} \underline{\mathrm{ho}}\mathrm{lim}\Big(\prod_\alpha \mathfrak{C}(U_\alpha) \rightrightarrows \prod_{\alpha\beta} \mathfrak{C}(U_{\alpha\beta}) \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} \prod_{\alpha\beta\gamma} \mathfrak{C}(U_{\alpha\beta\gamma}) \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} \dots\Big).$$

Model categories

These are categories with a **model structure**:

- Three distinguished families of morphisms: fibrations, cofibrations and **weak equivalences**

subject to some axioms:

1. All limits and colimits exist;
2. The families of morphisms are stable under retraction;
3. Two-out-of-three property for weak equivalences;
4. Left lifting property for fibrations and cofibrations;
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The crucial data is the choice of weak equivalences:
Out of a model category C, one can form its homotopy category Ho(C) with the following universal property: There is functor $C \to Ho(C)$ sending weak equivalences to isomorphisms.

Examples of model categories

Remark:

- There is in general no unique model structure;
- Given the class of w.e.s and another class of morphisms, the third one is specified by the left-lifting property.

1. Topological spaces with weak (or strong) homotopy equivalences as weak equivalences;
2. Simplicial sets with weak homotopy equivalences via geometric realization as weak equivalences;
3. Groupoids with equivalences of categories as weak equivalences;
4. Chain complexes of R-modules with quasi-isomorphisms as w.e.s.

Homotopy (co)limits

These are *nasty* things to define, but one can still find "formulas" to compute them.

Roughly speaking, one wants a "softer, more flexible" analogue of a (co)limit. Instead of being stable with respect to isomorphism, it should be stable with respect to weak equivalences.

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A rigorous general definition is not quite easy to digest. We will use existing **“recipes” to compute homotopy (co)limits** of interest.

Typically, such recipes are based on the fact that one can find **good replacements** (fibrant or cofibrant) of the diagrams of interest. Then computing the homotopy (co)limit of the original diagram reduces to computing the ordinary (co)limit for the replacement.

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First physical input:
Only gauge invariant information should be “measurable”.

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We have (at least) two reasonable options:

1. Pass to the **set of isomorphism classes** and describe functionals (automatically gauge invariant) on it, i.e. take the “brutal” quotient by gauge transformations.
2. Carry **configurations and equivalences altogether until the end** of your construction and then focus on gauge invariant observables.

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Advantages and disadvantages:

1. Typically the quotient is geometrically badly behaved and we lose descent (no way to glue isomorphism classes);
2. Configurations and equivalences together form nice geometric objects with good descent conditions, however it is harder to work with groupoids rather than just sets.

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Nice geometry is important:
Furthermore, the fact that one has nicer geometric structure is crucial especially for non-Abelian gauge theories, as non-linearities require a choice of functionals which are regular enough.

Field theory (w/o gauge) vs. gauge field theory

	no gauge	gauge
configurations	sets	groupoids
descent	gluing on the nose (lim)	gluing <u>up to gauge</u> (<u>ho</u>lim)
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We take this as a definition of **global gauge observables**:

$$\mathfrak{O}(M) \xleftarrow{\simeq} \underline{\mathrm{ho}}\mathrm{colim}\Big(\coprod_{\alpha} \mathfrak{O}(U_\alpha) \leftleftarrows \coprod_{\alpha\beta} \mathfrak{O}(U_{\alpha\beta}) \overset{\textstyle\leftarrow}{\leftleftarrows} \coprod_{\alpha\beta\gamma} \mathfrak{O}(U_{\alpha\beta\gamma}) \overset{\textstyle\leftleftarrows}{\leftleftarrows} \dots\Big)$$

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Net advantage: Complicated groupoids of gauge theory on arbitrary manifolds, however much easier on contractibles. *Understand global observables by looking at the easy groupoids on contractibles!*

The broad picture

Going one step higher in category theory:
We have seen gauge field theory as a generalization of field theory (without gauge), where sets of configurations (with their gluing conditions) are replaced by groupoids of configurations together with equivalences (with gluing conditions modified coherently).

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This approach can be made more general by going higher to 2-, 3-, ... categories. What one gets are higher stacks encoding suitable descent properties.

Examples:

- bundle gerbes;
- bundle gerbes with connective structure;
- higher analogues.

The groupoids of gauge theory on contractibles

G a (matrix) Lie group, $\mathfrak{g}$ its Lie algebra, U a contractible manifold.

Observations:

- Each principal G-bundle P over U is isomorphic to the trivial one;
- Connections are described just by gauge potentials $A \in \Omega^1(U;\mathfrak{g})$;
- A gauge transformations is just a smooth function $g \in C^\infty(U;G)$.
- The action of gauge transformations on connections is

$$\rho : C^\infty(U;G)\times\Omega^1(U;\mathfrak{g}) \longrightarrow \Omega^1(U;\mathfrak{g}), \;\; \rho(g,A) = g\,A\,g^{-1}+g\,\mathrm{d}\,g^{-1}.$$

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We obtain an **action groupoid**:

Objects: The set $\Omega^1(U;\mathfrak{g})$ of gauge potentials;

Isomorphisms: The set $C^\infty(U;G) \times \Omega^1(U;\mathfrak{g})$;

Action: $(A,g) : A \longrightarrow \rho(A,g)$;

Smoothness: It can be endowed with the structure of a Lie groupoid.

Changing perspective

It is not clear (at least to me) which is the correct category to model "functions on groupoids". However, one can present groupoids as **simplicial sets** (even simplicial manifolds for Lie groupoids). Then the functions on them form **cosimplicial algebras**.

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The simplicial set associated to the gauge groupoid on U:

$$\Omega^1(U;\mathfrak{g}) \leftleftarrows C^\infty(U;G)\times\Omega^1(U;\mathfrak{g}) \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; C^\infty(U;G)^{\times 2}\times\Omega^1(U;\mathfrak{g}) \;\overset{\longleftarrow}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longleftarrow}}}\; \cdots$$

This comes together with face maps ∂_i^n (displayed) and degeneracy maps ϵ_i^n (implicit):

$$\begin{aligned}
\partial_i^n : C^\infty(U;G)^{\times n}\times\Omega^1(U;\mathfrak{g}) &\longrightarrow C^\infty(U;G)^{\times n-1}\times\Omega^1(U;\mathfrak{g}),\\
(g_1,\dots,g_n,A) &\longmapsto (g_1,\dots,g_i\,g_{i+1},\dots,g_n,A);\\
\epsilon_i^n : C^\infty(U;G)^{\times n}\times\Omega^1(U;\mathfrak{g}) &\longrightarrow C^\infty(U;G)^{\times n+1}\times\Omega^1(U;\mathfrak{g}),\\
(g_1,\dots,g_n,A) &\longmapsto (g_1,\dots,g_i,e,g_{i+1},\dots,g_n,A).
\end{aligned}$$

Observables for gauge theory on contractibles

Consider the functor $C : \mathsf{Sets}^{\mathrm{op}} \to \mathsf{Alg}$ sending each set to the algebra of $\mathbb{C}$-valued functions on it. Applying this functor at each level of the simplicial set immediately provides a **cosimplicial algebra**:

$$C\big(\Omega^1(U;\mathfrak{g})\big) \rightrightarrows C\big(C^\infty(U;G) \times \Omega^1(U;\mathfrak{g})\big) \overset{\textstyle\rightarrow}{\rightrightarrows} \cdots ,$$

with coface maps $d_n^i = C(\partial_i^n)$ (displayed) and codegenaracy maps $e_n^i = C(\epsilon_i^n)$ (implicit in the notation).

For nice groupoids, say Lie groupoids, we may even choose smooth functions. This is crucial for non Abelian gauge theory, however not needed in the Abelian case.

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Cosimiplicial algebras are a model category, so we can *in principle* compute our homotopy colimits, however the model structure is not quite easy to handle!

Recapitulating...

Start Groupoids of gauge configurations on contractibles;

Goal Define observables on a generic manifold computing a homotopy colimit for a contractible cover.

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We propose the following approach:

1. On a **contractible** manifold, gauge configurations can be arranged into a nice and simple **action groupoid** (even smooth);
2. Passing to simplicial sets (or manifolds), we get a good handle on the categorical structure for "functions on groupoids";
3. In fact, by taking functions at each level of the simplicial set, one gets a cosimplicial algebra;
4. Unfortunately, this is not so easy to handle!

Abelian gauge theory

To circumvent the difficulties with cosimplicial algebras, we assume $G = U(1)$. Then:

1. The action by gauge transformations boils down to $\rho(A, g) = A + g^{-1}\,\mathrm{d}\,g$;
2. The simplicial set becomes in fact a simplicial Abelian group;
3. Instead of arbitrary $\mathbb{C}$-valued functions, we can take smooth $U(1)$-valued group homomorphisms;
4. The cosimplicial algebra is replaced by a cosimplicial Abelian group;
5. Via dual Dold-Kan, we end up in non-positively graded chain complexes of Abelian groups.

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With CHAIN complexes, you can compute like a CHAINpion"!

In fact, for this model category we have rather explicit formulas.

The relevant chain complexes

U contractible manifold of dimension m, $G = U(1)$.

For configurations we have the normalized Moore complex:

$$\begin{aligned} 0 \longleftarrow \Omega^1(U;\mathfrak{g})_0 &\xleftarrow{\delta} C^\infty(U;G)_1 \longleftarrow 0, \\ g \,\mathrm{d}\, g^{-1} &\longleftarrow\!\shortmid g. \end{aligned}$$

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For observables we have the conormalized Moore complex:

$$\begin{aligned} 0 \longleftarrow \Omega^m_{c,\mathbb{Z}}(U;\mathfrak{g}^*)_{-1} &\xleftarrow{\delta^*} \Omega^{m-1}_c(U;\mathfrak{g}^*)_0 \longleftarrow 0, \\ \mathrm{d}\,\varphi &\longleftarrow\!\shortmid \varphi. \end{aligned}$$

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Pairing: $\langle \chi \oplus \varphi, A \oplus g\rangle = \exp\Big(\int_U \big(A \wedge \varphi + \log(g)\chi\big)\Big).$

A mathematical "experiment" 1

For configurations:

1. Recall the non-neg. graded chain complex of Abelian groups for $U(1)$-gauge theory on contractible manifolds;
2. Take a manifold M and cover it by all its contractible open subsets;
3. This cover provides a diagram in chain complexes describing $U(1)$-gauge configurations on each contractible region of M;
4. Compute its homotopy limit to reconstruct global gauge configurations on M from local ones.

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Results:

- $\underline{\mathrm{ho}}\mathrm{lim}$ spits out a functor from $\mathsf{Man}^{\mathrm{op}}$ to chain complexes;
- On contractible manifolds the output of the homotopy limit is weakly equivalent to the input;
- The homotopy limit reproduces all bundle-connection pairs.

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- On contractible manifolds the output of the homotopy colimit is weakly equivalent to the input;
- The homotopy colimit detects all $U(1)$-bundle-connection pairs.

In summary...

- **Homotopical algebra is a good framework for gauge theory;**
- Global configurations arise via homotopy limit from coherent data subordinate to a cover by contractible open subsets;
- Dually, global observables can be obtained via homotopy colimits from observables on contractible open subsets;
- Advantage: Gauge theory is easy on contractible manifolds.
- Disadvantage: Homotopy (co)limits are not so easy to work out.

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- Dually, global observables can be obtained via homotopy colimits from observables on contractible open subsets;
- Advantage: Gauge theory is easy on contractible manifolds.
- Disadvantage: Homotopy (co)limits are not so easy to work out.

In the next part:

- To make the computation tractable, we take Abelian gauge theory;
- We determine global configurations up to gauge computing the $\underline{\mathrm{ho}}\mathrm{lim}$ we just discussed;
- We produce observables capable of detecting all configuration up to gauge computing the $\underline{\mathrm{ho}}\mathrm{colim}$ we just discussed.