ZMP

Modules over VOAs

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I Motivation

Space of fields $\mathcal{F} \supset V_H$: holomorphic fields

VOAs formalize this notion

+ OPE w/ crossing symmetry

Not all fields are holomorphic: E.g. spinless quasi-primary $\mathcal{O}$:

$$\langle \mathcal{O}(z)\, \mathcal{O}(w) \rangle = \frac{C}{|z-w|^{2\Delta}}$$: 2pt-fn not holomorphic.

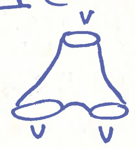
OPE + C.S. $\rightsquigarrow$ $V_H \times \mathcal{F} \longrightarrow \mathcal{F}$ $\rightsquigarrow$ VOA-modules

Idea: • understand irreducible V_H & V_A-modules (antiholomorphic)

• reconstruct $\mathcal{F}$ from these

II Modules over VOAs (Idea: keep all axioms of VOA that make sense)

Def (VOA) $V = \bigoplus_{n=N}^{\infty} V_n$ (fin. dim), $|0\rangle, \omega \in V$



$Y: V \longrightarrow \mathrm{End}\, V[[z^{\pm 1}]]$

$A \longmapsto \sum_{n\in\mathbb{Z}} A_n z^{-n-1} = Y(A,z)$

1) $A_n v = 0$ if n is large (truncation)

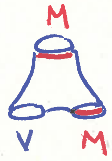
2) $Y(|0\rangle, z) = \mathrm{id}_V$, $\lim_{z\to 0} Y(A,z)|0\rangle = A$ (vacuum)

3) (locality)

4) $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n^V z^{-n-2}$ (Virasoro generators)

$L_0|_{V_n} = n \cdot \mathrm{id}_{V_n}$

Def (V-module) $M = \bigoplus_{\nu\in\mathbb{C}} M_\nu$ (fin. dim.)



$Y_M: V \longrightarrow \mathrm{End}\, M[[z^{\pm 1}]]$

1) (truncation)

2) (vacuum): $Y_M(|0\rangle, z) = \mathrm{id}_M$

3) ~~locality~~ (associativity)

4) $M_\nu = \{0\}$ if $\mathrm{Re}\,\nu < N$

$L_0^M|_{M_\nu} = \nu \cdot \mathrm{id}_{M_\nu}$

Rem Vir acts on M via $Y_M(\omega,z) = \sum_{n\in\mathbb{Z}} L_n^M z^{-n-2}$

Prop $\varphi \in M^\vee$ $A, B \in V$ $C \in M$

↖ graded dual

The following formal power series converge to the same rational fn on some domain:

$$\langle \varphi | Y_M(A,z)\, Y_M(B,w)\, C \rangle ,$$
$$\langle \varphi | Y_M(Y(A,z-w)B, w)\, C \rangle$$
$$\langle \varphi | Y_M(B,w)\, Y_M(A,z)\, C \rangle .$$

Easy example: • VOA over itself

• A : commutative associative unital alg ($\Rightarrow$ VOA)

M : module over A (as algebra) $\Rightarrow$ M is a module over A as a VOA-module

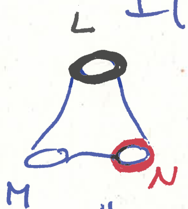
Def RepV : category of V-modules

objects: V-modules (M, N)

morphisms: $\mathrm{Hom}_V(M,N) = \{ f : M \to N$ lin. s.t. $f(Y_M(A,z)C) = Y_N(A,z) f(C) \}$

→ submodule → irreducible module

Def $M, N, L \in \mathrm{Rep}V$ The space of intertwining operators of type $\binom{L}{M\;N}$



$$I\binom{L}{M\,N} = \{ i : M \otimes_{\mathbb{C}} N \to L\{z\} \mid \cdot (\text{truncation}),\ \cdot \text{intertwiner} \}$$

↖ formal powerseries w/ complex exponents

" $Y_L(A,w)\, i(m\otimes n, z) = i(Y_M(A,w-z)m \otimes n, z) = i(m \otimes Y_N(A,w)n, z)$ " : intertwiner

→ 3pt conformal blocks ⇝ conformal blocks ⇝ correlators

Def $M, N \in \mathrm{Rep}\,V$ $\quad$ $\underline{M * N} \in \mathrm{Rep}\,V$ s.t. $\forall\, L \in \mathrm{Rep}\,V$ $\exists$ isomorphism

$$\Theta_L : I\binom{L}{M\ N} \longrightarrow \mathrm{Hom}_V(M*N, L) \quad \text{natural in } L.$$

($M*N$ is the representing object of the functor $\mathrm{Rep}\,V \to \mathrm{Vect}_{\mathbb{C}}$, $L \mapsto I\binom{L}{M N}$)

$\to$ Existence in general is unknown, in some cases ✓, $\rightsquigarrow$ monoidal str. on $\mathrm{Rep}\,V$.

III Example: Modules over the Heisenberg VOA ($\mathcal{F}_H$)

Let $\alpha \in \mathbb{C}$. Will define a module over $\mathcal{F}_H$ in 3 steps:

1) $\mathbb{C}^\alpha := \mathrm{span}\{|\alpha\rangle\}$: 1d rep. of $\mathfrak{b} = \mathrm{span}\{a_n, K \mid n \geq 0\}$

via $\quad a_n|\alpha\rangle = 0 \quad n>0 \qquad a_0|\alpha\rangle = \alpha|\alpha\rangle \qquad K|\alpha\rangle = |\alpha\rangle$

2) $N^\alpha := \mathrm{Ind}_{\mathfrak{b}}^{H} = \mathrm{span}\{\underbrace{a_{-n_1} \cdots a_{-n_k}|\alpha\rangle}_{=\,|\underline{n},\alpha\rangle} \mid k \geq 0 \ n_i > 0\}$ $\qquad \underline{n} \in (\mathbb{Z}_+)^k$

3) $Y_{N^\alpha}(J, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$ $\rightsquigarrow$ gives unique $\mathcal{F}_H$-module str. on N^α

Virasoro action on N^α: $\quad (\omega_\lambda = \frac{1}{2} J^2 + \lambda \partial J)$

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} :a_m a_{n-m}: - \lambda(n+1) a_n$$

$$L_0 |\underline{n}, \alpha\rangle = \Big[\underbrace{\alpha\big(\tfrac{\alpha}{2} - \lambda\big) + \sum_{i=1}^{k} n_i}_{\nu}\Big] |\underline{n}, \alpha\rangle$$ $\rightsquigarrow$ gives $\mathbb{C}$-gradation $N^\alpha = \bigoplus_{\nu \in \mathbb{C}} N^\alpha_\nu$
(actually just a shifted $\mathbb{Z}$-gradation)

<u>Rem</u> $N^0 \simeq \mathcal{F}_H$ as $\mathcal{F}_H$-modules

<u>Thm</u> 1) N^α is an irreducible $\mathcal{F}_H$-mod $\forall \alpha \in \mathbb{C}$

2) if N is an irreducible $\hat{\mathcal{F}}_H$-mod $\Rightarrow N \simeq N^\alpha$ for some $\alpha \in \mathbb{C}$

<u>Unitarity</u> 1) Define an inner product on N^α s.t. $a_n^+ = a_{-n}$ $(n \in \mathbb{Z})$

$$\Rightarrow a_0^+ = a_0 \Leftrightarrow \boxed{\alpha \in \mathbb{R}}$$

2) Ask: When is N^α a unitary Vir-rep?
$L_n^{N^\alpha +} = L_{-n}^{N^\alpha}$ $(n \in \mathbb{Z})$?

$$\Leftrightarrow \quad \lambda^*(n+1)\, a_n^+ = \lambda(-n+1)\, a_{-n} \Leftrightarrow \boxed{\lambda = 0}$$