Vertex algebras

main reference: [FBZ] Frenkel, Ben Zvi – Vertex algebras and algebraic curves

also: [S] Schottenloher – A mathematical introduction to CFT
[LL] Lepowsky, Li – Introduction to VOA's

Aim of this talk: introduce mathematical precise objects that implement the following features in a very natural way:

- state field correspondence
- n-pt functions, satisfying
- bootstrap conditions (h ~ u ...)
- OPE's with crossing symmetry
- choices of Virasoro actions

Definition 1: (Vertex algebra, basically [FBZ])

A Vertex algebra is a collection of data:

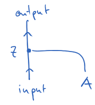
- a vector space V (space of states)
- a distinguished vector $|0\rangle \in V$ (vacuum vector)
- a linear operator $\partial : V \to V$ (translation operator)
- a linear operation $Y(\cdot, z) : V \longrightarrow \operatorname{End} V[[z^{\pm 1}]]$, each $A \in V$ to a field

$$Y(A,z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}$$

i.e. for every $v \in V$ $\exists N : A_n v = 0$ $\forall n \geq N$.

formal power series $\sum_{n\in\mathbb{Z}} A_n z^{-n}$ with coefficients $A_n \in \operatorname{End}(V)$ in the space of linear operators from V to V.

Recall from last time: Insertion of a state A at a point z gives an operator $Y(A,z)$, which depends holomorphically on z $\Rightarrow$ this is often called a field.



The operation Y is called vertex operator.

The above data are subject to the following axioms:

(i) $Y(|0\rangle, z) = \mathrm{Id}_V$ and $Y(A,z)|0\rangle \in V[[z]]$ (vacuum axiom)

Thus guarantees that we can evaluate $Y(A,z)|0\rangle$ in $z=0$ and we want furthermore $Y(A,z)|0\rangle\big|_{z=0} = A$.

(ii) $[\partial, Y(A,z)] = \frac{\partial}{\partial z} Y(A,z)$ and (translation axiom)
$\partial|0\rangle = 0$

The operator ∂ is completely defined in this way: $\partial(A) = A_{-2}|0\rangle$

most important axiom, allowing for bootstrap, OPE, ...

(iii) All fields $Y(A,z), Y(B,z)$ are local wrt each other, i.e

$$\exists N \geq 0 : (z-w)^N [Y(A,z), Y(B,w)] = 0 \quad \text{(locality axiom)}$$

In particular, this means that the commutator $[Y_A(z), Y_B(w)]$ is supported only on the diagonal $z=w$. An important example for such a power series (in two variables z,w) is the formal delta function $\delta(z-w) := \sum_{m\in\mathbb{Z}} z^m w^{-m-1}$, which has the important property that $A(z)\delta(z-w) = A(w)\delta(z-w)$ and in particular $(z-w)\delta(z-w) = 0$ (m.g. $(z-w)^{n-1}\partial_w^n \delta(z-w) = 0$)
It turns out that the delta function and its derivatives form a basis for all local $f \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$.
↑ some ring with 1

In order to make life easier, one often includes the following axiom:

(iv) In order to make things easier, one often requires V to be graded and bounded from below:

$V = \bigoplus_{z=K}^{\infty} V_z$ ← f.d. ; vertex op's "preserve this structure" and $\deg|0\rangle = 0$, $\deg \partial = 1$ (gradation axiom)

i.e. $\deg A_n = \Delta - n - 1$ for some conformal dimension $\Delta \in \mathbb{Z}$

Energy bounded from below!

gradation given by eigenvalues of dilatation operator L_0

Before we discuss the properties of vertex algebras in detail, we start with a very simple example, which nevertheless can be seen as a blueprint for more interesting VA's. We will make this statement precise later.

Example 2: (Heisenberg vertex algebra) in physics, this is called the free boson

Let $\mathbb{C}[z,z^{-1}]$ denote the ring of Laurent polynomials (i.e. $f = \sum$ finitely many f_z's non-zero

We define a Lie algebra structure on $\mathcal{H} := \mathbb{C}[z,z^{-1}] \oplus \mathbb{C}\cdot K$ the commutation relations

$$[a_n, a_m] = n\cdot\delta_{n,-m}\cdot K$$
and
$$[z, a_n] = 0 \quad \forall n \in \mathbb{Z}.$$

$\mathcal{H}$ is a central extension of the commutative Lie alg. $\mathbb{C}[z,z^{-1}]$ wrt the 2-cocycle $c(f,g) = \operatorname{Res}_{z=0}(f \cdot \partial_z g)$.

include **Example 1:** (commutative VA)

Let A commutative, associative unital algebra and fix a derivation ∂ on A.

Then, $Y(a,z)(b) := \sum_{n\geq 0} \frac{z^n}{n!}(\partial^n a)b = (e^{z\partial}a)b$

defines a commutative vertex algebra, i.e. $[Y(a,z), Y(b,w)] =$
We need both commutativity and associativity of A in order to show this commutativity property:

$$Y(a)Y(b)c = a(bc) = (ab)c = (ba)c = b(ac) = Y(b)Y(a)c$$

the commutation relations $[a_n, a_m] = n\cdot\delta_{n,-m}\cdot K$ and $[Z, a_n] = 0 \quad \forall n \in \mathbb{Z}$.

$\mathcal{H}$ is a central extension of the commutative Lie alg. $\mathbb{C}[z,z^{-1}]$ wrt the 2-cocycle $c(f,g) = \mathrm{Res}_{z=0}(f\cdot\partial g)$.

We "need" both commutativity and associativity of "A" in order to show this commutativity property:

$$Y(a)Y(b)c = a(bc) = (ab)c = (ba)c = b(ac) = Y(b)Y(a)c$$

We will see in what sense Vertex algebras can be seen as generalizations of commutative associative algebras (with unit and derivation)

$\mathcal{H}$ is called the (rank 1) Heisenberg algebra.

Now, let $V = \mathcal{F}_{\mathcal{H}}$ be the Fock space representation of $\mathcal{H}$, i.e.

$\mathcal{F}_{\mathcal{H}} := \mathbb{C}[J, \partial J, \partial^2 J, \dots]$, where for $n \in \mathbb{N}$

- $a_n \cdot \partial^k J = n\cdot\delta_{k,n-1}$ — Annihilators act by derivation
- $a_{-n}\cdot\partial^k J = \partial^{n-1} J\, \partial^k J$ — Creators act by left multiplication
- $a_0 \cdot \partial^k J = 0$

compare with [FBZ]: $a_n = b_n$; $\partial^{k-1}J = b_{-k}$

Creators $a_n^{\dagger} := a_{-n}\cdot(-)$ create a state $\partial^{n-1}J$ from the vacuum $|0\rangle := 1$, while annihilators $a_n := a_n\cdot(-)$ kill every state by repeated application.

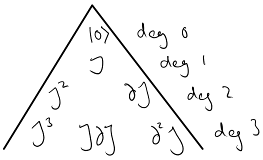
Brief mathematical description of $\mathcal{F}_{\mathcal{H}}$: Let b denote the positive part of $\mathcal{H}$, generated by the a_n, $n \geq 0$ and K. By $\mathbb{C}_0$, we denote the 1-dim rep of b given by $Z.1 = 1$ and zero else.

Then $\mathcal{F}_{\mathcal{H}}$ is induced by $\mathbb{C}_0$: $\mathcal{F}_{\mathcal{H}} = \mathcal{U}(g) \otimes_{\mathcal{U}(b)} \mathbb{C}_0$. We close the notation $|0\rangle = |\mathbb{1}\rangle$, $J = a_{-1}.|\mathbb{1}\rangle$

The notation was choosen in a way, such that the action of the translation operator ∂ is (up to a factor) obvious: $\partial(\partial^{k-1}J) = k\cdot\partial^k J$. Also, ∂ acts by derivation on the comm. algebra $\mathcal{F}_{\mathcal{H}}$.

The monomial degree $\deg(\partial^{j_1}J \dots \partial^{j_k}J) := \sum_{i=1}^{k} j_i$ defines a $\mathbb{Z}_+$-gradation on $\mathcal{F}_{\mathcal{H}}$.

Now, we want to define $Y(\,.\,,z)$ on $\mathcal{F}_{\mathcal{H}}$. The Fock space looks like this:



The value $Y(|0\rangle, z) = \mathrm{Id}$ is determined by the vacuum axiom, so the next step is $Y(J,z)$. We will see that THIS IS ALL WE HAVE TO DO!

Set $\boxed{Y(J,z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1}}$ (1)

An easy calculation shows, that $Y(J,z)$ is indeed a field of conformal dimension $\Delta = 1$.

Before we continue defining the other vertex operators, we want to check that the locality of $Y(J,z)$ with respect to itself:

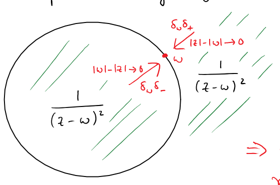
$$[Y(J,z), Y(J,w)] = \sum_{n,m\in\mathbb{Z}} [a_n, a_m] z^{-n-1} w^{-m-1}$$

$$\overset{\text{commutator relations}}{=} \sum_{n\in\mathbb{Z}} n\cdot z^{-n-1} w^{n-1} = \partial_w \delta(z-w).$$

But we have seen that $(z-w)^2 \cdot \partial_w \delta(z-w) = 0$. ✓

Interpretation of $\partial_w\delta(z-w)$: Split $\partial_w\delta(z-w) = \underbrace{\sum_{n\leq 0} n z^{-n-1} w^{n-1}}_{\partial_w\delta_-} + \underbrace{\sum_{n>0} n z^{-n-1} w^{n-1}}_{\partial_w\delta_+}$ whose

fix w!



$\Rightarrow$ $\partial_w\delta_-$ and $-\partial_w\delta_+$ can be seen as boundary values of the rational functions $\frac{1}{(z-w)^2}$ defined in the interior and exterior of the circle $z = w$. Their difference $\delta_w\delta(z-w)$ is supported only in the point $z = w$.

As a next step, we want to define what Y does on the descendants $\partial^k J$, $k \geq 1$.

As the notation indicates, a reasonable definition is

$$Y(\partial^k J, z) = \frac{1}{k!}\partial_z^k Y(J,z)$$

These vertex operators are local as well which can easily seen by differentiating the locality condition of $Y(J,z)$:

$$(z-w)^2 [Y(J,z), Y(J,w)] = 0 \quad \xrightarrow[\cdot(z-w)]{\partial_z} \quad (z-w)^3 [\partial_z Y(J,z), Y(J,w)] = 0$$

and so on (induction).

Now, the next step in the pyramid is $Y(\partial J^2, z)$. One is tempted to define $Y(\partial J^2, z) = Y(\partial J, z)^2$

BUT THIS IS NOT A FIELD! We would have $Y(\partial J^2, z)_{-2} = \sum_{k+l=0} a_k a_l = \sum_{k\in\mathbb{Z}} a_k a_{-k}$

$$= \sum_{k<0} a_{-k}^{\dagger} a_{-k} + \underbrace{\sum}_{\text{becomes infinite!!}}$$

Solution: Rearrange the sum, such that annihilators always act before creators.

We define the normally ordered product of $Y(J,z)$ with itself by $:Y(J,z)Y(J,z): := \sum_{n\in\mathbb{Z}} :a_k a_l: z^{-n-2}$,
where
$$:a_k a_l: := \begin{cases} a_l a_k, & l=-k, k\geq 0 \\ a_k a_l, & \text{else}. \end{cases}$$
$\Rightarrow$ This gives a field.

In general, we define for two fields $A(z), B(w)$ $\quad :A(z)B(w): = \underbrace{A(z)_+}_{\sum_{n\geq 0} A_n z^n} B(w) + B(w)\underbrace{A(z)_-}_{\sum_{n<0} A_n z^n}$.

Lemma 1: (Lemma 2.3.4 & Lemma 2.2.3 in [FBZ])

(i) $A(z), B(z)$ fields of conformal dim Δ_A, Δ_B $\Rightarrow$ $:A(z)B(z):$ fields of conf. dim $\Delta_A+\Delta_B$

(ii) $A(z), B(z), C(z)$ mutually local fields $\Rightarrow$ $:A(z)B(z):, C(z)$ mutually local. $\square$

Corollary 1: All fields $Y(\partial^{j_1}J \cdots \partial^{j_2}J, z) = \prod_i (j_i!)^{-1} :\partial_z^{j_1} Y(J,z) \cdots \partial_z^{j_2} Y(J,z):$ are mutually local.

Similarly, from $[\partial, Y(J,z)] = \partial_z Y(J,z)$ we can conclude that this holds for all $A\in \mathcal{F}_H$
and from $Y(J,z)|0\rangle = J + z \cdots$ —"— $\square$

$\Longrightarrow$ $\mathcal{F}_H$ is a vertex algebra!

The following theorem makes precise in what sense $\mathcal{F}_H$ was a blueprint.

Theorem 1: (Weak reconstruction theorem) Suppose that you have a vector space V that is build such as $\mathcal{F}_H$, but with more than one generator allowed. Call the generators J_i.
← i.e. more pyramids

If there are $|0\rangle \in V$, $\partial \in \mathrm{End}(V)$, $Y(J_i,z) \in \mathrm{End}V[[z^{\pm 1}]]$,
such that all the axioms are fulfilled, then
$$Y(J_{i_1}^{j_1} \cdots J_{i_q}^{j_q}, z) = \frac{1}{j_1! \cdots j_q!} :\partial_z^{j_1} Y(J_{i_1},z) \cdots \partial_z^{j_q} (J_{i_q},z):$$

defines a vertex algebra structure on V.

Finally, we show that there is a canonical Virasoro action on $\mathcal{F}_H$

Definition 2. (vertex operator algebra)
A vertex algebra V is said to be a VOA of central charge $c\in\mathbb{C}$, if there is a non-zero conformal vector $\omega \in V_2$, such that the Fourier coefficients L_n^V of the corresponding vertex operator
$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n^V z^{-n-2} =: T(z) \qquad (2)$$
satisfy the defining relations of the Virasoro algebra with central charge c, and moreover, $T = L_{-1}^V$ and $L_0^V\big|_{V_n} = n \cdot \mathrm{Id}_{V_n}$. //

Lemma 2: In case of the Heisenberg vertex algebra $V=\mathcal{F}_H$, we have a 1-parameter family of conformal structures, given by
$$\omega_\lambda = \frac{1}{2} J^2 + \lambda \cdot \partial J$$
Then, ω_λ is a conformal vector with central charge $c_\lambda = 1 - 12\lambda^2$. $\square$

Associativity and Operator product expansion

We want to make precise in what sense vertex algebras can be seen as generalizations of commutative and associative algebras.

Theorem 2 (commutativity and associativity)
For all $A, B, C \in V$, the formal power series

(1) $Y(A,z)Y(B,w) \in \mathrm{End}V[[z^{\pm}, w^{\pm 1}]]$ converges in the domain $|z| > |w|$ to an operator valued function.

(2) $Y(B,w)Y(A,z) \in \mathrm{End}V[[w^{\pm}, z^{\pm 1}]]$ —"— $|w| > |z|$ —"—

(3) $Y(Y(A,z-w)B,w) \in \mathrm{End}V[[(z-w)^{\pm}, w^{\pm 1}]]$ —"— $|w| > |z-w|$ —"—

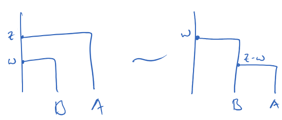
All three fields can be analytically continued to the same meromorphic operator-valued function on $\mathbb{C}^2$ with poles at $z=0, w=0, z=w$. $\square$

Remark:

(i) The fact that this holds for (1) and (2) is called the commutativity property of VA's.
It is actually equivalent to the locality axiom.

(ii) The fact that (1) & (3) can be analytically continued to the same meromorphic function is called the associativity property. It needs all axioms of a vertex algebra, including the identities

- $Y(a,z)|0\rangle = e^{z\partial} a$



In physics literature one often sees this

this explains why

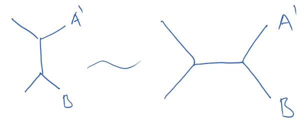
(**) The fact that (1) & (2) can be analytically continued to the same meromorphic function is called the associativity property. It needs all axioms of a vertex algebra, including the identities

$e^{z\partial} := \sum_{n\geq 0} \frac{z^n}{n!}\partial^n$

- $Y(a,z)|0\rangle = e^{z\partial} a$
- $Y(A,z+w) = e^{w\partial} Y(A,z) e^{-\partial w}$ ← this explains why ∂ is called the translation operator
- $Y(A,z)B = e^{z\partial} Y(B,-z)A$ (skew-symmetry)

following from translation and vacuum axiom.

In physics literature one often sees this



Both pictures are imprecise since they are 1-dim.

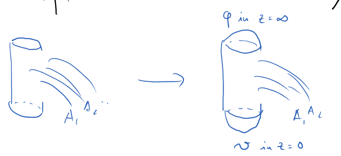
The operator product expansion is then (by abuse of notation) written as

$$Y(A,z)Y(B,w)C \overset{\text{not really!}}{=} Y(Y(A,z-w)B,w)C = \sum_{n\in\mathbb{Z}} \frac{Y(A_{(n)}\cdot B,w)}{(z-w)^{n+1}} C.$$

n-pt functions and bootstrap

Definition: (n-pt function) For $v\in V$, $\varphi\in V^*$, $A_i\in V$, we define the n-pt function as

$$\langle \varphi, Y(A_1,z_1)\dots Y(A_n,z_n)v\rangle.$$



The vacuum axiom tells us that it is enough to know them for $v = |0\rangle$ (and $\varphi = \langle 0| = |0\rangle^*$ if $V = \bigoplus_n V_n$ ← fin. dim. and $V^\vee = \bigoplus_n V_n^*$)

This gives a formal power series in $\mathbb{C}[[z_1^{\pm 1},\dots,z_n^{\pm 1}]]$.

Theorem 3 (bootstrap condition) For any $\varphi\in V^*$ the power series $\langle\varphi, Y(A_1,z_1)\dots Y(A_n,z_n)|0\rangle$ is an expansion of a series $f_{A_1,\dots,A_n}(z_1,\dots,z_n) \in \mathbb{C}[[z_1,\dots,z_n]][(z_i-z_j)^{-1}, i\neq j]$, s.t.

(↑ only positive power; ↖ finite sum)

(1) $f_{A_1,\dots,A_n}$ is invariant under permutation of arguments.

(2) For $i<j$:

$$f_{A_1,\dots,A_n}(z_1,\dots,z_n) = f_{\underbrace{Y(A_i,z_i-z_j)A_j}_{\text{OPE!}},A_1,\dots,\hat{A}_i,\dots,\hat{A}_j,\dots,A_n}(z_1,\dots,\hat{z}_i,\dots,z_j,\dots,z_n),$$

where $(z_i-z_j)^{-1}$ has to be replaced by its expansion $\sum_{k\geq 0} z_i^{-k-1}z_j^k$.

Thus, we can compute the n-pt function in terms of an n-1 pt function!

(3) $1\leq j\leq n$:

$$\partial_{z_j} f_{A_1,\dots,A_n}(z_1,\dots,z_n) = f_{A_1,\dots,\partial A_j,\dots,A_n}(z_1,\dots,z_n)$$

□