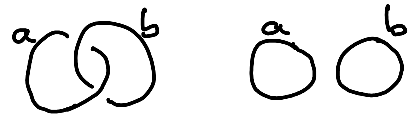
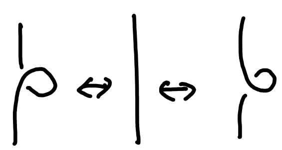
ZMP seminar: 3d TQFT and modular functors

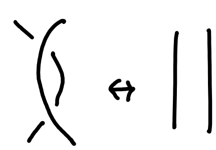
1) Ribbon categories and link invariants
2) Modular tensor categories and Reshetikhin–Turaev TQFTs
3) Modular functors

1) Ribbon categories and link invariants

Two links in $\mathbb{R}^3$ with colours 

Reidemeister moves

I) 

II) 

III) 

Idea: braided cat. with duals

$\mathcal{C}$: cat. e.g. vect : f.d. $\mathbb{C}$-v.sp. & lin. maps

$\text{vect}^{\mathbb{Z}_N}$: $\mathbb{Z}_N$-graded f.d. $\mathbb{C}$-v.sp. and grade pres. lin. maps

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_{N-1}$$

tensor 1) functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

e.g. vect : $U, V \mapsto U \otimes_{\mathbb{C}} V$

$\text{vect}^{\mathbb{Z}_N}$: —"— s.th. $(U \otimes V)_a = \bigoplus_{x \in \mathbb{Z}_N} U_x \otimes V_{a-x}$

2) isom. $U \otimes (V \otimes W) \xrightarrow{\alpha_{UVW}} (U \otimes V) \otimes W$ <u>associator</u>

e.g. vect : $x \otimes (y \otimes z) \mapsto (x \otimes y) \otimes z$ "$\alpha = id$"

$\mathrm{vect}^{\mathbb{Z}_N}_{\omega}$ -- $\mapsto \underbrace{\omega(|x|, |y|, |z|)}_{\omega : \mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{C}^*} \cdot$ ---

two ways $U(V(WX)) \xrightarrow{\sim} ((UV)W)X$ "pentagon"

e.g.

$a, b, c, d \in \mathbb{Z}_N$: $\omega(b,c,d)\,\omega(a,b+c,d)\,\omega(a,b,c)$
$= \omega(a+b,c,d)\,\omega(a,b,c+d)$

3) unit $\mathbb{1}$, isom. ... e.g. $\mathbb{1} = \mathbb{C}$, resp. $\mathbb{C}_0$

<u>braided</u> : isom. $U \otimes V \xrightarrow{c_{UV}} V \otimes U$

e.g. vect : $x \otimes y \mapsto y \otimes x$

$\mathrm{vect}^{\mathbb{Z}_N}$: $x \otimes y \mapsto \underbrace{\beta(|x|, |y|)}_{\beta : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{C}^*}\, y \otimes x$

+ hexagon conditions

e.g.

$a, b, c \in \mathbb{Z}_N$ $\omega\, \beta(a+b, c)\, \omega = \beta(a,c)\, \omega\, \beta(b,c)$
$\omega\, \beta(a, b+c)\, \omega = \beta(a,b)\, \omega\, \beta(a,c)$

e.g. : $\omega = 1$ $\beta(a,b) = e^{\frac{2\pi i}{N} ab}$

<u>duals</u> : for all obj $U \in \mathcal{C}$: U^*, $U^* U \xrightarrow{ev} \mathbb{1}$ $UU^* \to \mathbb{1}$
$\mathbb{1} \xrightarrow{coev} UU^*$ $\mathbb{1} \to U^* U$

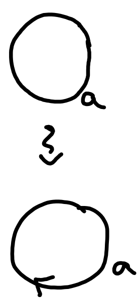
+ conditions

e.g. vect : dual v.-sp. $U^* \otimes U \longrightarrow \mathbb{C}$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & U \otimes U^* \arrow[d, "\wr"] \\ & \mathrm{Hom}(U,U) \end{tikzcd}, \quad 1 \mapsto \sum u_i \otimes u_i^*$$

$\mathrm{vect}^{\mathbb{Z}_N}_{1,\beta}$: $(U^*)_a := (U_{-a})^*$

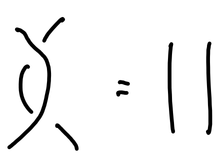
How to compute

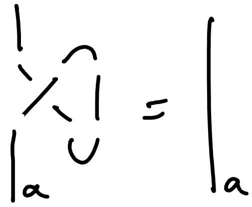


$$\mathbb{1} \longrightarrow a \otimes a^* \longrightarrow \mathbb{1}$$

e.g. vect : $a = V$, ... $= \dim V$

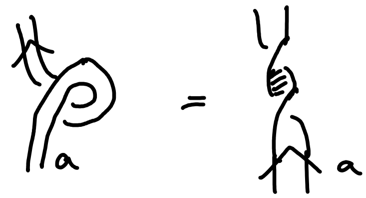
Reidemeister ?

E.g. II  : $c \circ c^{-1} = \mathrm{id}$ ✓

I  vect : ✓

$\mathrm{vect}^{\mathbb{Z}_N}_{1,\beta}$: $\beta(x,x) = 1$ ↯

$\boxed{\text{framed, oriented links}}$

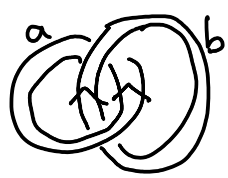


<u>ribbon cat.</u> br. tens. cat, duals + <u>twist</u> : iso $U \xrightarrow{\theta_U} U$
+ cond.

e.g vect : $\theta_U = id_U$.

$vect^{\mathbb{Z}_N}_{\omega,\beta}$: $U \longrightarrow U$
$x \longmapsto \beta(|x|,|x|) \cdot x$

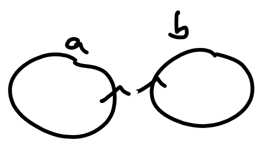
Compute:



vect : $a = U$, $b = V$ ... $= \dim U \cdot \dim V$

$vect^{\mathbb{Z}_N}_{1,\beta}$: $a = \mathbb{C}_x$, $b = \mathbb{C}_y$

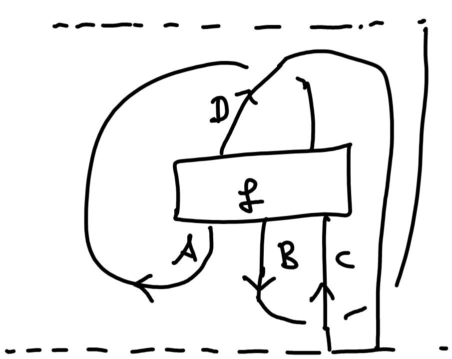
$\ldots = \beta(x,y)\cdot\beta(y,x) = e^{\frac{4\pi i}{N}xy}$



vect : $\dim U \dim V$

$vect^{\mathbb{Z}_N}_{1,\beta}$: 2

More general ribbon tangle:



$f : A^* B^* C \longrightarrow D A^*$

evaluates to morph

$C D^* \longrightarrow B$ in $\mathcal{C}$

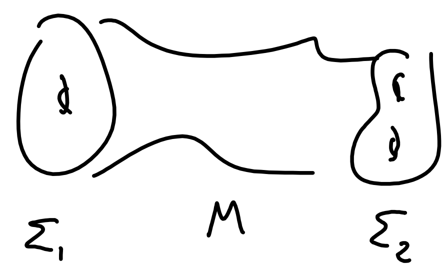
Thm (Reshetikhin, Turaev '90) "evaluation in $\mathcal{C}$ only depends on isotopy class of ribbon tangle in $\mathbb{R}^2 \times [0,1]$."

2) MTC and RT TQFTs

3d TQFT :

closed 3mf $M \rightsquigarrow Z(M) \in \mathbb{C}$

closed 2mf $\Sigma \rightsquigarrow Z(\Sigma)$, a $\mathbb{C}$-v.sp. (fin. dim.)



$Z(\Sigma_1) \xrightarrow{Z(M)} Z(\Sigma_2)$

$\Sigma_1 \quad M \quad \Sigma_2$

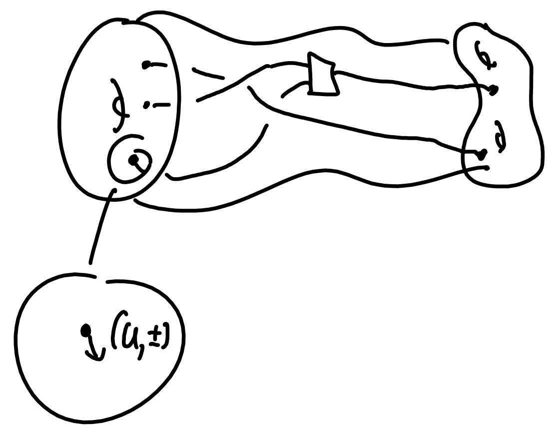
$Z(\Sigma_1 \sqcup \Sigma_2) \simeq Z(\Sigma_1) \otimes Z(\Sigma_2)$

$\times \longmapsto$ flip of tens. fact.

$\Sigma_2 \sqcup \Sigma_1$

<u>Def</u> A 3d TQFT is a symmetric monoidal functor $\mathrm{Bord}_3 \longrightarrow \mathrm{Vect}$

Want also ribbon gr. :



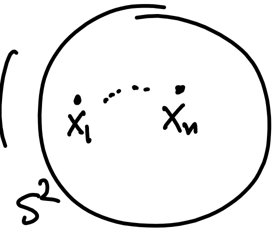
RT constr. : semisimple ribbon cat$/\mathbb{C}$ + (F) + (N)

<u>modular tensor cat.</u>

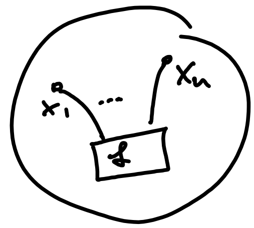
(F) finite # of simple obj S_i $(i \in I)$

e.g. vect : $S = \mathbb{C}$ $|I| = 1$

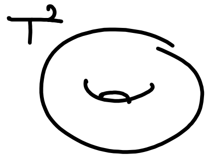
$\mathrm{vect}^{\mathbb{Z}_N}$: $S_x = \mathbb{C}_x$, $x \in \mathbb{Z}_N$, $|I| = N$

Properties : $Z\Big($ 

$\Big) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X_1 \otimes \cdots \otimes X_n)$

$Z\big(\, B^3$ 

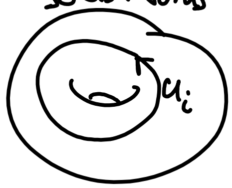
$\big) \longleftarrow\!\shortmid \; f$

$Z\big($ 

$\big) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, L)$

$$L = \bigoplus_{i \in I} U_i \otimes U_i^*$$

solid torus

$Z\Big($ 

$\Big) \longleftarrow\!\shortmid \; \mathrm{coev}_{U_i}$

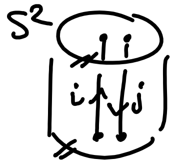
$$Z(\Sigma_g, X_1 \cdots X_n) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X_1 \cdots X_n L^{\otimes g})$$

Compute a bit :

$$Z(S^2; i, j) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, U_i \otimes U_j)$$

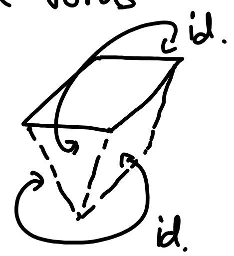
1-dim : $U_j \simeq U_i^*$
0-dim : else

$$Z(\Sigma \times S^1) = \mathrm{tr}_{Z(\Sigma)}\, \mathrm{id}_\Sigma = \dim Z(\Sigma)$$

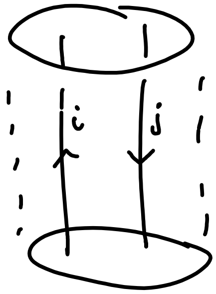
$Z\Big($ 

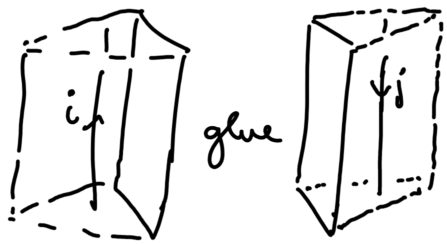
$\Big) = \delta_{ij}$

solid torus



$S^2\times S^1$



= 

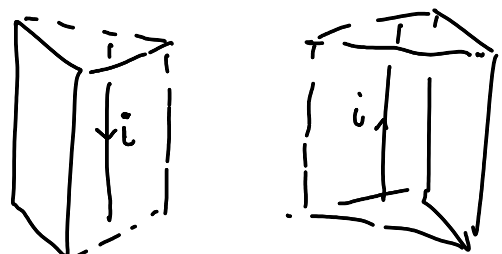
(*)

$\emptyset \longrightarrow T^2 \longrightarrow \emptyset$

apply Z

δ_{ij} $\qquad \mathbb{C} \longrightarrow Z(T^2) \longrightarrow \mathbb{C}$

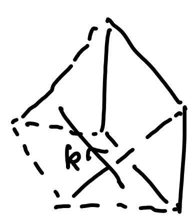
Now

$\sum_i Z\Big($ 

$\Big) = \mathrm{id}_{Z(T^2)}$ (**)

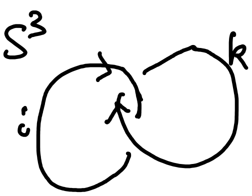
$T^2 \longrightarrow \emptyset \longrightarrow T^2$

1) Condition (N)

act on 

to get

$\emptyset \longrightarrow T^2$

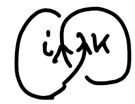
$Z(\uparrow) = \sum_i Z\Big($ S^3 

$\Big) Z(\quad)$

$\overset{!!}{S_{ik}}$

Thus

(N) $|I|\times|I|$-Matrix S must be invertible

Now $S_{ik} = Z(S^3) \cdot \Big($ribbon inv. of  $\Big)$

e.g. vect : $i = k = 1$, $\;= 1$ MTC ✓

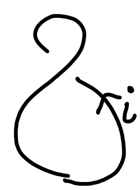
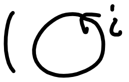
$\mathrm{vect}^{\mathbb{Z}_N}_{1,\beta}$: $S_{xy} = Z(S^3) \cdot e^{\frac{4\pi i}{N} xy}$

invertible for N odd
not inv. for N even

2) Surgery

S^3

cut torus around L,
glue back after S-transf.

$Z($ surg. along L $) = \sum_{i \in I} Z\big(S^3$  $\big) \cdot Z(S^3)$
$\cdot$ rib.inv.()

Thm (Reshetikhin-Turaev '91, Turaev '94)
A MTC $\mathcal{C}$ gives sym. mon. fun.

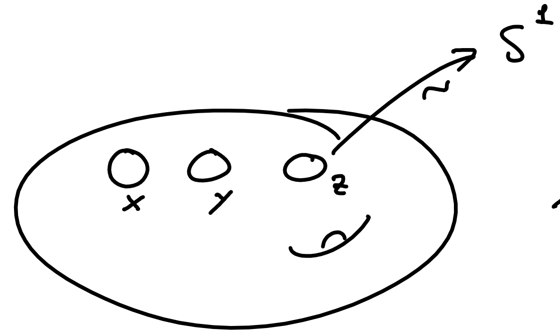
(Bord. with $\mathcal{C}$-col. ribbons + ...) $\longrightarrow$ vect

cancel anomaly

3) Modular functor

$\mathcal{C}$: finite s.si cat.

Cat $\mathcal{M}_{\mathcal{C}}$: obj:



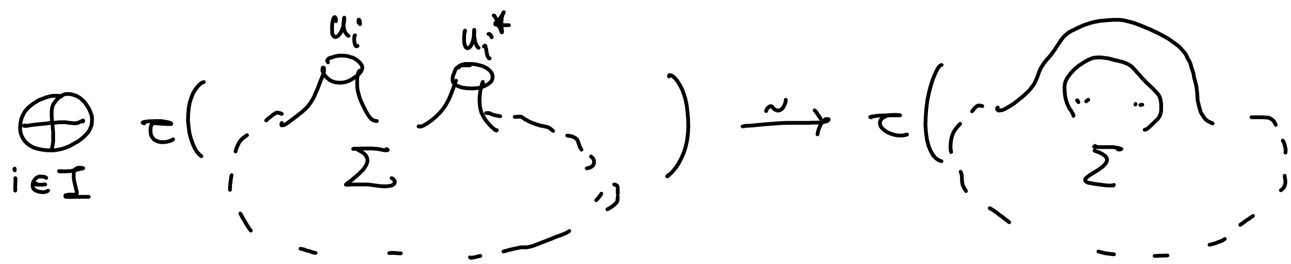
$x, y, z \in \mathcal{C}$

morph: homeo comp. with bnd. param / isotopy

e.g. $\mathrm{Hom}(\Sigma, \Sigma)$: mapping class group of Σ

Def:
A modular functor for $\mathcal{C}$ consists of

- a sym. mon. functor $\tau : \mathcal{M}_{\mathcal{C}} \longrightarrow \mathrm{vect}$
- glueing maps $G(\Sigma)_{a,b}$



+ conditions.

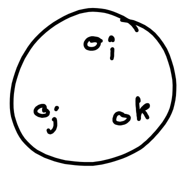
Thm. (Moore-Seiberg '89, Turaev '94)
A MTC $\mathcal{C}$ gives a m.f. for $\mathcal{C}$

Idea: $\tau(\Sigma) = Z(\Sigma)$

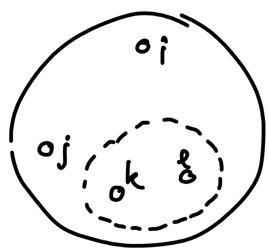
$\tau(\Sigma \xrightarrow[\sim]{f} \Sigma') = Z(\Sigma \xrightarrow{f} \Sigma \times [0,1] \xleftarrow{id} \Sigma)$

Thm. (Bakalov–Kirillov '01)
A mf. for $\mathcal{C}$ plus one extra assumption (duals) gives an MTC on $\mathcal{C}$.

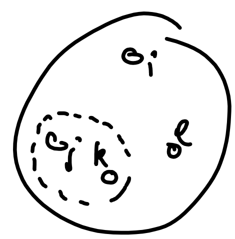
Idea tens. prod.



assoc.



vs



braid.