VOAs, Zhu's C_2-algebra and Higgs branches

Based on: Beem, Rastelli, VOAs, Higgs branches & modular differential equations 1707.07679

see also: Arakawa, Associated varieties and Higgs branches (a survey). 1712.01945

outline
1. Jan's talk: Higgs branch
2. Pedro's talk: VOA from $\mathcal{N}=2$ SCFT
3. Lóránt's talk: Zhu's C_2 algebra
4. Conjecture, consequences and example (Tobias)

$\mathcal{N}=2$ SCFTs in 4d

From Jan's talk: simpler algebra inside the full OPE algebra:

"Higgs branch" chiral ring R_H

- Consider a particular type of short operator:

$$\hat{\mathcal{B}}_{\underset{SU(2)_R}{R_1}}^{\uparrow}(x)\,\hat{\mathcal{B}}_{R_2}^{\uparrow\,\text{l.w.}}(0) \sim \mathcal{O} + \hat{\mathcal{B}}_{R_1+R_2}^{\uparrow} + \dots \qquad \text{Form a ring}$$

$x \to 0 \quad \sim$ multiplication in ring.

$\Rightarrow$ commutative, associative, Poisson algebra.
(imprint of hyperkähler geometry is P.B. associated to the complex symplectic forms built out of the appropriate linear comb. of hyperkähler forms)

- "Geometrization" of R_H:

"Higgs branch" $\mathcal{M}_H = \operatorname{spec} R_H$, $\quad R_H = \mathbb{C}[\mathcal{M}_H]$
algebraic variety

Assumes Higgs branch is always affine complex algebraic variety
$\to$ take this as a def. for the purpose of the talk, but it's a conjecture based on physics

$\Rightarrow R_H$ is a reduced algebra, i.e. no nilpotent elements
(conjecture)

Jam also mentioned a physics def

Fact: $\mathcal{N}=2$ SCFTs have w/ moduli space of supersymmetric vacua (found by minimizing pot. of theory)

↳ "Higgs branch": subspace of the moduli space of vac. that breaks $SU(2)_R$ but preserves $U(1)_r$.

Lagrangian theories: give vev to hyper mult. scalar.

It's a holomorphic symplectic variety

Lore: $\mathcal{M}_H$ is parametrized by vevs of Higgs branch ops

so R_H is the coordinate ring of the Higgs branch $\mathcal{M}_H$ (ring of functions on algebraic variety coincides w/ R_H)

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Recall: coord. ring of complex affine variety is a finitely generated reduced $\mathbb{C}$-algebra from which the variety can be uniquely reconstructed as a scheme $\mathcal{M} = \mathrm{Spec}\, R$.

$\mathbb{C}$-algebra is a ring containing complex numbers as a subring

$\mathrm{Spec}(R) = \{ P \subset R \mid P \text{ is } \underbrace{\text{prime ideal}}_{a,b \in R,\ ab \in P \Rightarrow a\in P,\ b\notin P \text{ or } b\in P,\ a \notin P} \}$ endowed with the

$\forall x \in I \quad \forall r \in R, \ x \cdot r,\ r \cdot x \in I$

Zariski topology and a structure sheaf making it into a scheme

Holomorphic symplectic variety: Kähler, simply connected, admits holomorphic symplectic form

Every 4d $\mathcal{N}=2$ SCFT comes equipped w/ an intricate algebraic invariant, the associated VOA:

Pedro: $\exists$ map χ : $\mathcal{N}=2$ SCFTs$/_{\text{marginal defs}}$ $\longrightarrow$ $\overset{\text{VOA}}{\mathcal{V}}$

ops in short irreps $\longmapsto$ $\underline{Sl(2)}$ primaries

• Obtain by cohomological construction
↳ "Higgs branch" chiral ring operators are in cohomology of $\mathbb{Q}$
$\underbrace{\text{Higgs branch chiral ring}}_{\hat{\mathcal{B}}_R}$

• From 4d OPE selection rules one can show that $\hat{\mathcal{B}}_R$ are strong generators of $\underline{\mathcal{V}}$. (and Vir. prim)
↳ sl(2) primary that is not normal ordered product
⇑ cannot appear as non sing term in OPE

• But there can be, and are, many other strong generators of $\mathcal{V}$, e.g. Stress tensor $T(z)$ is not of "Higgs" type, different short.

Conjecture (Expectation) $\mathcal{V}$ is strongly finitely generated

Note: Strong generators of $\mathcal{V}$ & their singular operator prod coeffs completely classify the VOA; we always remove null states, passing to the unique simple quotient of the Verma module.

Q1 Are the operators descending from $\hat{\mathcal{B}}_R$ special (Can we reconstruct $\mathcal{R}_H$ from $\mathcal{V}$? (and therefore $\mathcal{M}_H$)

In VOA there is no obvious distinction between these and other ops. T has to go!

Recall we want to recover a commutative associative Poisson algebra... We have seen one:

mode shift

$a^{math}(m) = a^{phys}_{(-h_a+1+m)}$

Lónámt's talk   Zhu's C_2-algebra

$C_2[V] := \text{span}\{ a_{\underbrace{-h_a-1}} \phi \ , \ a, \phi \in V \}$ $\lim_{z\to 0} a(z)\Omega = a_{-1}$

vector space

math labels: -2 math labels $-m-1$

(stress that like Pedro I use labeling: $a \longmapsto Y(a,z) = \sum_m a_{(m)} z^{\overbrace{-m-h_a}}$)

$\lim_{z\to 0} a(z)\Omega = a_{-h_a}$

Note $a_{(-h_a-m)}\Omega \longleftrightarrow \partial^m a$ by state/operator map

$C_2[V]$ is roughly the space of normally ordered composites that include derivatives (can include non-derivative stuff if set equal to derivative stuff by null relation)

Theorem [Zhu] $R_V := V/C_2(V)$ is a commutative associative Poisson algebra w/ Poisson bracket $\{a,b\} := a_{-h_a+1}b$

Product $a *_0 b := a_{-h_a} b$

(should read $\{\bar{a},\bar{b}\} := \overline{a_{-h_a+1} b}$ w/ $\bar{a}$ image of $a\in V$ in R_V) & product $a \cdot b = a_{-h_a} b$

Higgs branch chiral ring ops are always in R_V (shown from selection rules)

If V is strongly finitely generated then R_V has a simple description as the space of polys in the strong generators modulo the ideal induced by $\underbrace{\ }_{null}$ relations

$$\mathcal{R}_{\mathcal{V}} := \mathcal{V}/C_2(\mathcal{V})$$

$R_{\mathcal{H}} = \mathcal{H}/\mathcal{V}_+$ can show $C_2(\mathcal{V}) \subseteq \mathcal{V}_+$

This is not enough: $T(z)$ has to go but $L_{-2}\Omega \notin C_2(\mathcal{V})$

Also: C_2-algebra is not reduced, while $\mathcal{R}_{\mathcal{H}}$ is!

$$\mathcal{R}_{\mathcal{H}} = \mathcal{R}_{\mathcal{V}}/\mathcal{I}_+ \quad , \quad \mathcal{I}_+ \subseteq \mathrm{Nil}(\mathcal{R}_{\mathcal{V}})$$

Conjecture [Beem Rastelli] $\mathcal{I}_+ = \mathrm{Nil}(\mathcal{R}_{\mathcal{V}})$

$$\mathcal{R}_{\mathcal{H}} = (\mathcal{R}_{\mathcal{V}})_{\mathrm{red}}$$

$\Leftarrow \mathcal{R}_{\mathcal{V}}$ quotiented by its nilradical $\mathrm{Nil}(\mathcal{R}_{\mathcal{V}})$ (i.e. all nilpotent elements).

Def: [Arakawa] Associated variety of VOA $\mathcal{V}$:

$$X_{\mathcal{V}} := \mathrm{Spec}(\mathcal{R}_{\mathcal{V}})_{\mathrm{red}}$$

$\|$ (by other conj.)

$\mathcal{M}_{\mathcal{H}}$ Higgs branch is associated variety.

Def $\mathcal{V}$ is lisse (or C_2-cofinite) if $\mathcal{R}_{\mathcal{V}}$ is finite dimensional.

(This is necessary cond. for rationality.)

Equivalently $\dim X_{\mathcal{V}} = 0 \Rightarrow$ trivial Higgs branch

Rmk
Only theories w/o Higgs branch are rational (small portion of $\mathcal{N}=2$ SCFTs)

Example Lee-Yang: (Gioam Lónánt's talk)
$\uparrow \chi$

"(A_1, A_2) Argyres-Douglas" theory - interesting interacting SCFT w/ no conventional Lagrangian description.

Virasoro element is the single strong generator of V

$L_{-2}|0\rangle := \omega$

R_V has single generator $[\omega] = \omega$
$\parallel$
$V/C_2(V)$

At generic c $\quad R_V \cong \mathrm{span}\{\underbrace{(L_{-2})^k}|0\rangle, \; k=1,2,\dots\}$

multiplication $[\omega]^{k_1} *_0 [\omega]^{k_2} = [\omega]^{k_2}$ $\quad [\omega]^k$
induced by NO

Ring: multiplication of polys in one variable

$R_V = \mathbb{C}[[\omega]]$ $\Rightarrow$ freely generated ring

Poisson Bracket $\{[\omega], [\omega]\} = L_{-1}L_{-2}|0\rangle = L_{-3}|0\rangle \sim 0$

$\Rightarrow$ trivial

Take $c = -\frac{22}{5}$ for Lee Yang

$N = L_{-4}|0\rangle - \frac{3}{5} L_{-2}^2|0\rangle$ $\qquad J_c = \{L_{-1}|0\rangle, N\}$

$V = M_c / J_c$ Verma module

$\Rightarrow$ $\omega^2 \sim 0$ in $V/C_2(V)$ and so:

$R_V = \mathbb{C}[\omega]/\langle \omega^2 \rangle$ $\Rightarrow$ not reduced

$\Rightarrow$ Finite dimensional dim $R_V = 2$
(spanned by 1 and ω)

$(R_V)_{red}$ is trivial $\Rightarrow$ trivial Higgs branch
$\shortparallel$ $X_V = \{pt\} = \mathrm{Spec}(\mathbb{C})$
$\mathbb{C}$, generated by 1

- VOAs gen only by ω can only correspond to $\mathcal{N}=2$ SCFTs if $\omega^m \sim 0$ in $V/C_2(V)$

s.t. $(R_V)_{red}$ is trivial (ω has to go!)

Tobias' talk:

$$\mathcal{N} = \left(L[-4] - \frac{5}{3} L[-2]^2\right)|0\rangle \in O_q(V)$$

$$\left(D^2 + \frac{11}{3600} E_4(q)\right) Ch_0(q) = 0 \qquad D = q\frac{d}{dq}$$

E_4 ↳ Eisenstein series

Ch_0 = "Schur limit" of "superconformal index"

$\Rightarrow$ MLDE for Index.

Recall this was C_2-cofinite theory ($\Leftrightarrow$ no Higgs branch)

Non-trivial Higgs branches:

Def

A V-module is called ordinary if it is a positive energy representation on which L_0 acts semisimply and each L_0-eigenspace is finite-dim. s.t.

$$Ch_M(q) = tr|_M \left(q^{L_0 - \frac{c}{24}}\right)$$ is well defined.

Def [Arakawa Kawasetsu] strongly finitely generated VOAs (R_V finitely generated) whose associated varieties have finitely many symplectic leaves are called quasi-lisse.

(lisse $\Rightarrow$ quasi-lisse)

this is true of Higgs branches $\Rightarrow$ VOAs should be quasi-lisse
(1 leaf!) // (known conjecture holds + finitely gen)

X_V is Poisson variety so it has finite partition

$$X_V = \bigcup_{k=0}^{\sim} X_k$$ where X_k is a smooth analytic Poisson variety.

thus for any point $x \in X_k$ there is well defined symplectic leaf through it.

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<u>theorem</u> [Arakawa Kawasetsu] Let V be a quasi-lisse vertex algebra and M a <u>ordinary</u> V-module then Ch_M satisfies a modular linear diff eq.

(say words on what index is)

$\Rightarrow$ "Schur index" of $N=2$ SCFT satisfies mlde

$\quad$ ||

vacuum character of VOA

T is nilpotent in R_V $\leadsto$ T^k can give null but not guaranteed

$\hookrightarrow$ guarantees mlde exists, not necessarily of the order T could give.

Note VOA is $\frac{1}{2}\mathbb{Z}$ graded, MLDE we saw had E_{2k} modular forms, but now only $\Gamma^0(2)$ subgroup of $PSL(2,\mathbb{Z}) = \Gamma$ is relevant $\Rightarrow$ twisted modu. forms
$\Gamma^0(2) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ b = 0 \bmod 2 \}$.

Schur index transforms as a vector-valued (quasi-) modular form of weight zero under $PSL(2,\mathbb{Z})$ or $\Gamma^0(2)$

<u>Example</u>

$\mathcal{N}=2$ SCQCD: $SU(2)$ gauge theory + 4 $\square$ flavors

$\|$

Affine Kac Moody $SO(8)_{-2}$

$J^{A=1,\dots,\dim(SO(8))=28}_{-1}\,\Omega \in \mathcal{V}$

$j^A \in R_{\mathcal{V}}$ Poisson bracket $\{j^A, j^B\} = f^{AB}{}_C\, j^C$ ← str-const of $SO(8)$

$(j *_0 j)\big|_{\mathbb{1}} = \underbrace{\omega}_{\text{sugawara}}(k_{2d}+h^\vee)$, $\omega^2=0$

$(j*_0 j)\big|_{\underline{35}_{s,c,v}} = 0$

$$(R_{\mathcal{V}})_{red} = \overbrace{S(\mathfrak{g}_{\mathbb{C}})}^{\text{symmetric algebra}} / \mathcal{I}_2$$

$\mathfrak{g} = d_4$

$\mathcal{I}_2$ defined as $\mathrm{sym}^2(\mathrm{adj}) = (2\,\mathrm{adj}) \oplus \mathcal{I}_2$

Joseph ideal

$X_{\mathcal{V}}$ = Minimal nilpotent orbit of $\mathfrak{g}_{\mathbb{C}}$