Outline

1 Recap CFT
2 VOAs ⇝ second seminar today + block 2
3 Modular invariance ⇝ next time
4 Further comments

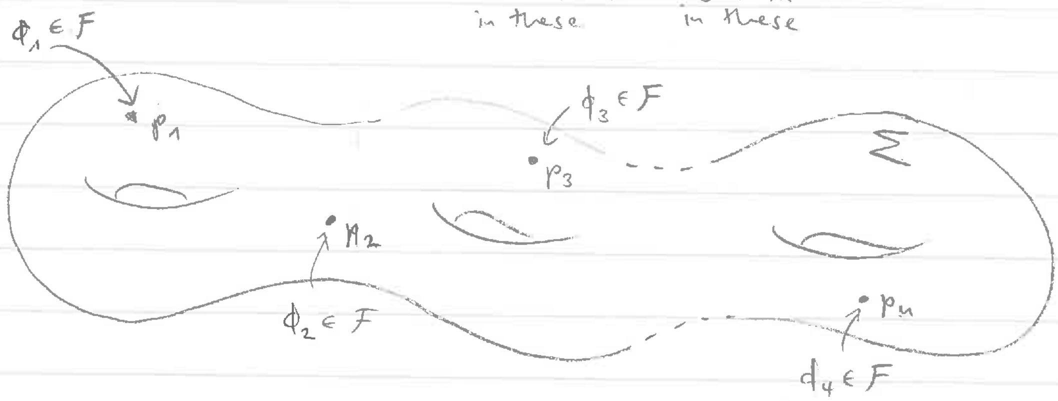
Aim: Motivate the next talks

1) Recap

A 2d CFT is (heuristically) a collection of maps (correlation functions)

$$\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle_\Sigma : \underset{\substack{\uparrow \\ \text{multilinear} \\ \text{in these}}}{\mathcal{F}^{\otimes n}} \times \underset{\substack{\uparrow \\ \text{smooth} \\ \text{in these}}}{\Sigma^n} \setminus \{p_i \neq p_j\} \longrightarrow \mathbb{C}$$

choice of local coordinates



here:

- Σ is a surface with a choice of conformal structure
  let's say compact, oriented
- $\mathcal{F}$ a vector space with a $\underset{\substack{\uparrow \\ \text{chiral} \\ \text{(holomorphic)}}}{Vir} \oplus \underset{\substack{\uparrow \\ \text{antichiral} \\ \text{(antiholomorphic)}}}{Vir}$ action
  (field space)

  N.B. "State-Field correspondence" $\leftrightarrow$ punctures $\underset{\substack{\uparrow \\ \text{conformal map}}}{\hat{=}}$ holes

These maps should satisfy consistency conditions wrt. cutting/gluing, and transform covariantly under conformal transformations.

Special case of cutting/gluing: There exists a way of colliding two points into one, and this operation is associative:

"operator product expansion"

OPE:

as $z_i \to z_j$ (but with finite radius of convergence)

$$\langle \dots \phi_i(z_i)\,\phi_j(z_j) \dots \rangle = \sum_k c_{ijk}\,(z_i - z_j)^{-h_i - h_j + h_k}\,(\bar{z}_i - \bar{z}_j)^{-\bar{h}_i - \bar{h}_j + \bar{h}_k}\,\langle \dots \phi_k(z_j) \dots \rangle$$

2) Vertex Operator Algebras

- OPE is a map $\mathcal{F} \times \mathcal{F} \longrightarrow \overline{\mathcal{F}}$.
- Consider the subspaces of $\mathcal{F}$:

$$\mathcal{F}_{hol} = \ker \bar{L}_{-1}$$
$$\mathcal{F}_{antihol} = \ker L_{-1}$$

N.B.: $\phi \in \mathcal{F}_{hol} \Rightarrow \bar{L}_{-n}\cdot\phi = 0 \ \forall n$ because:
- $\bar{L}_{-2}\cdot\phi \overset{Vir}{\sim} [\bar{L}_1, \bar{L}_{-1}]\cdot\phi$
- $\bar{L}_{-(n+1)}\cdot\phi \underset{Vir}{\sim} [\bar{L}_{-n}, \bar{L}_{-1}]\cdot\phi$ $(n \geq 2)$

- Then associativity of the OPE inspires us to say

$$\boxed{\text{"} \mathcal{F} \text{ is a module over } \mathcal{F}_{hol} \times \mathcal{F}_{antihol} \text{"}}$$

$\longrightarrow$ try to model $\underset{\overset{\shortparallel}{V}}{\mathcal{F}_{hol}}$ and $\underset{\overset{\shortparallel}{\bar{V}}}{\mathcal{F}_{antihol}}$ mathematically:

__Def__: A vertex operator algebra (VOA) is a graded vector space

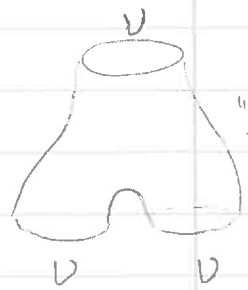
$$V = \bigoplus_{n=-\infty}^{\infty} V_n$$

together with a linear map

formal power series

$$V \longrightarrow (\operatorname{End} V)[[z, z^{-1}]]$$
$$a \longmapsto Y(a,z) = \sum_{n \in \mathbb{Z}} a(n)\, z^{-n-1}$$

Vertex operator of a ("field")



"$\mathcal{F}_{hol} \times \mathcal{F}_{hol} \to \mathcal{F}_{hol}$"

and distinguished elements $|0\rangle \in V_0$ (vacuum) and $\omega \in V_2$ (Virasoro element).

stress-energy tensor

NB: Vacuum axioms pictorially:

These data satisfy conditions, among which:

- $|0\rangle$ is the vacuum: $Y(|0\rangle, z) = 1$, $\lim_{z \to 0} Y(a,z)|0\rangle = a$

- $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ generates a copy of Vir acting on V.

  also: $$L_0 a = n a \quad \text{for } a \in V_n,$$
  $$Y(L_{-1}a, z) = \frac{d}{dz} Y(a,z)$$

- Jacobi identity, or equivalently locality:
  for all a, b there exists $N \in \mathbb{N}$ s.t.
  $$(z-w)^N \left[Y(a,z), Y(b,w) \right] = 0$$

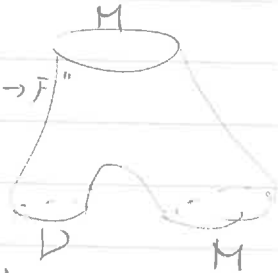
  N.B.: This implies in particular associativity, which was what we wanted from the OPE ...

Def: A module or representation of a VOA is a graded vector space

$$M = \bigoplus_{s \in \mathbb{C}} M_s \quad \text{typically trivial for almost all } s.$$

together with a linear map

$$V \longrightarrow (\mathrm{End}\, M)[[z, z^{-1}]]$$
$$a \longmapsto Y_M(a,z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$$



satisfying all the axioms of a VOA which still make sense.

Subtlety: Jacobi identity necessary, locality is not enough here.

Field space F: If this is the right definition, F should now have an action not only of Vir $\oplus$ Vir, but of $V_{hol} \oplus V_{antihol}$.

A semisimple CFT (e.g. rational CFT) has a fieldspace which is a direct sum of simple modules of $V_{hol} \oplus V_{antihol}$:

$$\boxed{F = \bigoplus_{i,j} (M_i \otimes M_j)^{\oplus Z_{ij}}} \quad Z_{ij} \in \mathbb{N}_0$$

N.B. Each of the M_i is also a Vir-representation through the action of ω

3) Modular invariance

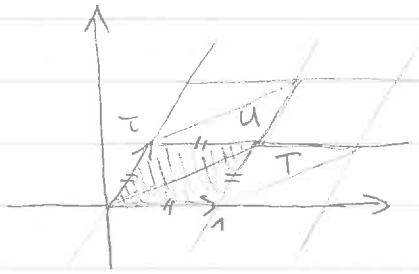
Idea: Let's find some simple constraints on $\mathcal{F}$!

0-pt amplitude (non-normalised correlation function) for Σ = torus

Let's construct torus partition function and impose that it must be conformally invariant!

"covariance" becomes "invariance" for 0 punctures

Fact: Every (Riemannian) torus is conformally equivalent to a torus of the form $\mathbb{C}/_{\mathbb{Z}+\tau\mathbb{Z}}$ for some $\tau \in \mathbb{H}$:



But: some of those τ give equivalent conformal structures

= complex structure in 2d

Fact:

$\tau, \tau' \in \mathbb{H}$ are conformally equivalent if and only if

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad \text{for some} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

modular group

$SL(2,\mathbb{Z})$ is generated by

$T: \tau \longmapsto \tau + 1$ (horizontal twist)

$S: \tau \longmapsto -\frac{1}{\tau}$ (composition TUT where U is vertical twist)

The moduli space of inequivalent complex structures on the torus is thus:

$$\boxed{\mathcal{M}_{1,0} = \mathbb{H}/_{SL(2,\mathbb{Z})}}$$

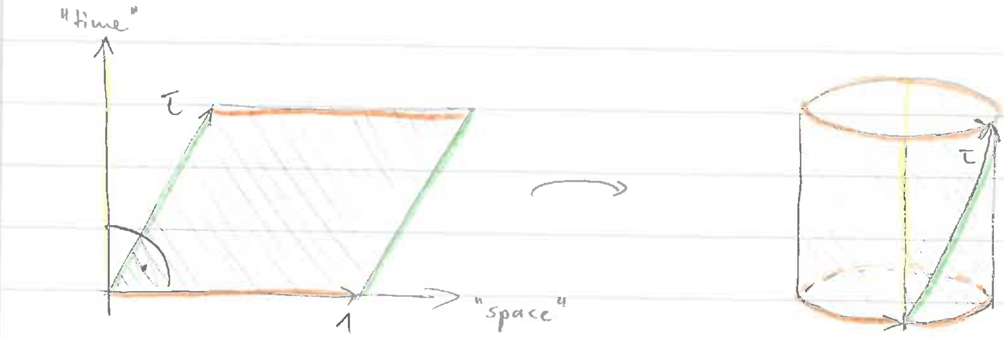
genus, punctures

Teichmüller space (complex structures modulo biholomorphism isotopic to the identity)

mapping class group (isotopy classes of biholomorphisms)

Now build torus partition function as a function on $\mathbb{H}$ and then demand invariance under $SL(2,\mathbb{Z})$ action.

Torus partition function



Prescription for partition function:

- associate with the field space $\mathcal{F}$.
- generator of space translations (rotations of the cylinder)

  $P = 2\pi (L_0 - \bar{L}_0)$
- generator of "time" translations (translations along infinite direction of cylinder)

  $H = 2\pi \left(L_0 + \bar{L}_0 - \frac{c}{12} \right)$

  generator of dilations on $\mathbb{C}$

  correction acquired with the transformation from $\mathbb{C}/\mathbb{Z}$ to $\mathbb{C}$ (cylinder)

  (not quite Möbius!)
- identify and : Finite shift given by map

  $$\mathcal{O}_\tau : \mathcal{F} \longrightarrow \mathcal{F}$$
  $$\phi \longmapsto e^{i \operatorname{Re}\tau P} \cdot e^{-\operatorname{Im}\tau H} = e^{i\pi\left[(\tau+\bar\tau)(L_0-\bar L_0)+(\tau-\bar\tau)(L_0+\bar L_0-\frac{c}{12})\right]}$$
  $$= e^{2\pi i\left[\tau\left(L_0-\frac{c}{24}\right)-\bar\tau\left(\bar L_0-\frac{c}{24}\right)\right]}$$
- impose boundary condition and sum over all histories: trace!

$$\boxed{Z(\tau) = \operatorname{tr}_{\mathcal{F}} \mathcal{O}_\tau = \operatorname{tr}_{\mathcal{F}}\left(q^{L_0-\frac{c}{24}}\, \bar{q}^{\bar L_0-\frac{c}{24}}\right)}$$

here: $q = e^{2\pi i \tau}$ $\quad \bar q = e^{-2\pi i \bar\tau}$ $\quad \tau \in \mathbb{H} \longrightarrow |q| < 1$

Questions:
- Why should $Z(\tau)$ be a well-defined number? Does $\operatorname{tr}_{\mathcal{F}}$ make any sense?
- If it does, what consequences do the conditions

  $$Z(\tau+1) \doteq Z(\tau) \doteq Z\left(-\tfrac{1}{\tau}\right)$$

  have?

Characters

Actually, with the presumed form of the field space $\mathcal{F}$ we can also write

$$Z(\tau) = \sum_{i,j} Z_{ij} \chi_i(\tau) \chi_j(\bar{\tau})$$

character of VOA-module:

$$\chi_j(\tau) = \mathrm{tr}_{M_j} q^{L_0 - \frac{c}{24}} = q^{h_j - \frac{c}{24}} \sum_{k=0} \left(\dim (M_j)_{h_j+k}\right) q^k$$

highest weight of $M_j = \bigoplus_{k \in \mathbb{N}} (M_j)_{h_j+k}$

Let's see how $SL(2,\mathbb{Z})$ acts on characters!
The T-transformation is easy:

$$\chi_j(\tau+1) = \mathrm{tr}_{M_j} e^{2\pi i (L_0 - \frac{c}{24})} q^{L_0 - \frac{c}{24}} = \overbrace{e^{2\pi i (h_j - \frac{c}{24})}}^{T_{jj}} \chi_j(\tau)$$

$$\chi_{\bar{j}}(\bar{\tau}+1) = \mathrm{tr}_{M_{\bar{j}}} e^{-2\pi i (\bar{L}_0 - \frac{c}{24})} \bar{q}^{\bar{L}_0 - \frac{c}{24}} = e^{-2\pi i (\bar{h}_j - \frac{c}{24})} \chi_{\bar{j}}(\bar{\tau})$$

The requirement $Z(\tau+1) = Z(\tau)$ already gives a simple constraint on $\mathcal{F}$:

$$\boxed{Z_{ij} \neq 0 \implies h_i - \bar{h}_j \in \mathbb{Z}} \qquad (*)$$

Let's look at an example: At central charge $c = \frac{1}{2}$ we find the Ising VOA with 3 simple modules:

M_0 (vacuum), $M_{\frac{1}{2}}$, $M_{\frac{1}{16}}$

The characters can be calculated: (analytic expression available)

$$\chi_0 = \chi_{\text{Verma}} - \text{"something"} = q^{-\frac{c}{24}} \left(1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \dots\right)$$

$$\chi_{1/2} = q^{\frac{1}{2} - \frac{c}{24}} \left(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \dots\right)$$

$$\chi_{1/16} = q^{\frac{1}{16} - \frac{c}{24}} \left(1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + \dots\right)$$

$P(n) = \#$ partitions of n

Note: It is easy to see that $\chi_{\text{Verma}} = q^{-\frac{c}{24}} \sum_{n=0}^{\infty} P(n) q^n =$

$$= q^{-\frac{c}{24}} \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k}\right) = q^{-\frac{c}{24}} \prod_{k=1} \left(\sum_{n=0}^{\infty} q^{kn}\right) : \text{ converges for } |q| < 1! \ (\tau \in \mathbb{H})$$

→ so all characters converge, since their coefficients are smaller than $P(j)$ because they belong to a quotient of the Verma module

→ $Z(\tau)$ makes sense!

Now <u>magic</u> happens! these characters transform into each other under the other modular transformation

$$\chi_i\left(-\tfrac{1}{\tau}\right) = \sum_j S_{ij}\,\chi_j(\tau) \qquad i,j = M_0, M_{\frac12}, M_{\frac{1}{16}}$$

where S is an involutory matrix

$$S = \frac12 \begin{pmatrix} 1 & 1 & \sqrt2 \\ 1 & 1 & -\sqrt2 \\ \sqrt2 & -\sqrt2 & 0 \end{pmatrix} = S^{-1} = S^T$$

for which $T^{-1} = T^* = T^\dagger$

which together with $T_{ij} = e^{2\pi i (h_j - \frac{c}{24})}\delta_{ij}$ forms a representation of $SL(2,\mathbb{Z})$ ($S^4 = 1$ & $S^2 = (ST)^3$). What does this mean for modular invariance of $Z(\tau)$? $Z(-\frac{1}{\tau}) = (S\chi)^T Z S\chi = \chi^T(S^T Z S)\chi = \chi^T(S^{-1}ZS)\chi \overset{!}{=} \chi^T Z \chi = Z(\tau)$

$$\boxed{Z(\tau) \text{ modular invariant } \Longrightarrow [Z,S] = 0 \overset{\checkmark}{=} [Z,T]}$$

Check the remaining condition! Z must be of the form

this is usually also imposed as an axiom.

unique vacuum

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & m \end{pmatrix}$$

no offdiagonal elements due to (*)

So

$$SZS = \frac14 \begin{pmatrix} 1 & n & m\sqrt2 \\ 1 & n & -m\sqrt2 \\ \sqrt2 & -n\sqrt2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & \sqrt2 \\ 1 & 1 & -\sqrt2 \\ \sqrt2 & -\sqrt2 & 0 \end{pmatrix} = \frac14 \begin{pmatrix} 1+n+2m & \cdots \\ 1+n-2m & \\ \vdots & \end{pmatrix}$$

$$\Longrightarrow \quad 1 + n \pm 2m = \begin{cases} 4 \\ 0 \end{cases} \quad \Longrightarrow \quad n = m = 1$$

$$\Longrightarrow \quad \mathcal{F}_{\text{Ising}} = M_0 \otimes M_0 \oplus M_{\frac12} \otimes M_{\frac12} \oplus M_{\frac{1}{16}} \otimes M_{\frac{1}{16}}$$ is the only possible combination (for a field space with chiral & antichiral VOA $\mathcal{V}_{\text{Ising}}$)

<u>Note</u>: • Typically there are several different opions (e.g. 3state Potts, a nondiagonal version of minimal model $\mathcal{M}(6,5)$), but $Z = 1$ is always possible (diagonal modular invariant, "Cardy case")
• The story with $SL(2,\mathbb{Z})$ repr. on the character generalises! (see next semin...

4) Further comments

- We just looked at the simplest constraints. The condition of modular invariance should typically also be imposed for higher genus and any number of punctures.

- At n-punctures the 0-pt amplitude $Z = \sum_{ij} Z_{ij} \underset{\text{0pt conformal blocks}}{\chi_i \bar{\chi}_j}$ generalises to

$$\langle \dots \rangle = \sum_{ij} d_{ij} \underset{\text{conformal n-pt. blocks}}{c_i \bar{c}_j}$$

To describe those conformal blocks one can study intertwiners of VOA-modules.

- We have not yet discussed cutting / gluing constraints.
  $\longrightarrow$ they relate correlators of different Σ to one another

- Surprise : It turns out that

$$N_{ijk} = \sum_n \frac{S_{in} S_{jn} (S^{-1})_{kn}}{S_{0n}} \in \mathbb{N} \qquad \text{(Verlinde formula)},$$

This formula relates
N_{ijk} – fusion of VOA-modules (genus 0 info)
S_{ij} – modular properties (higher genus info)