The Higgs branch of 4d $\mathcal{N}=2$ SCFTs

Plan: 1) Reminder: - 4d $\mathcal{N}=2$ superconformal algebra
- state - field correspondence and OPE in $d>2$

2) The Higgs chiral ring and the Higgs branch

3) Example: $\mathcal{N}=2$ $SU(2)$ SCQCD with $N_f = 2N_c$

1) A Lie_super algebra is a graded vector space $g = g_0 + g_1$ with a super Lie bracket $[\cdot,\cdot]: g \times g \to g$ satisfying certain axioms.

The 4d $\mathcal{N}=2$ superconformal algebra is called $g = su(2,2|2)$, where

* $g_0 = so(4,2) \oplus su(2)_R \oplus u(1)_r$

generators $\{M_{\mu\nu}, P_\mu, K_\mu, D\}$ $\{R_3, R_\pm\}$ $\{r\}$

labels $(\underbrace{j, \bar{j}}_{\in \frac{1}{2}\mathbb{N}_0}, \underset{\in \mathbb{R}}{\Delta})$ $\quad \underset{\in \frac{1}{2}\mathbb{N}_0}{R}$ $\quad \underset{\in \mathbb{R}}{r}$

* g_1 generated by

$$Q^A_\alpha \in \left(\tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) \qquad \alpha, \dot\alpha \in \{1,2\}$$
$$\tilde{Q}_{\dot\alpha A} \in \left(0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right) \qquad A \in \{1,2\}$$
$$S^\alpha_A \in \left(\tfrac{1}{2}, 0, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right)$$
$$\tilde{S}^{\dot\alpha A} \in \left(0, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

A superconformal primary is a state $|\Lambda\rangle$, for which

1) $\rho(K)|\Lambda\rangle = \rho(S)|\Lambda\rangle = \rho(\tilde{S})|\Lambda\rangle = \rho(R_+)|\Lambda\rangle = 0$

2) $\rho(D)|\Lambda\rangle = \Delta|\Lambda\rangle$, $\Delta \in \mathbb{R}$.

Representations of g are built by acting with $M, P, Q, \tilde{Q}, R_-$.

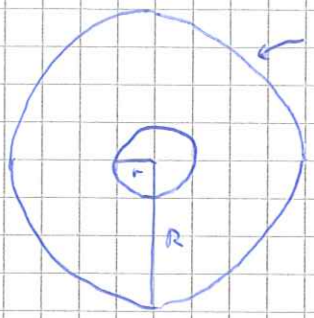
Shortening conditions: Example $\hat{B}_R$.

primary: $|\Lambda\rangle = |j = \bar{j} = r = 0, \Delta = 2R\rangle \Leftrightarrow \rho(Q^2_\alpha)|\Lambda\rangle = \rho(\tilde{Q}_{\dot\alpha 1})|\Lambda\rangle = 0$

state space

<u>state-field correspondence (heuristically)</u>: Let $|\Phi\rangle \in \mathcal{H}$

and $D|\Phi\rangle = \Delta|\Phi\rangle$



$|R^{-\Delta}\Phi\rangle$

finite dilatation

$$\left(\frac{R}{r}\right)^{D} \in \mathrm{End}(\mathcal{H})$$

$$\longleftrightarrow \qquad \left(\frac{R}{r}\right)^{D}\left(|R^{-\Delta}\Phi\rangle\right) = |r^{-\Delta}\Phi\rangle$$

spherical shell

The state $|\Phi\rangle$ is defined by local data, because r can be arbitrarily small. This can be modelled by a local operator → "field"

<u>OPE</u>:

$$\mathcal{O}_1(x)\,\mathcal{O}_2(0) \quad = \quad \sum_a c_a(x)\,\mathcal{O}_a(0)$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

$$\underbrace{\mathcal{O}_1(x)\,\mathcal{O}_2(0)|0\rangle}_{\in\mathcal{H}} \underset{\mathcal{H}\text{ is complete}}{=} \sum_{\underset{\text{some basis}}{a}} c_a(x)\,|\lambda_a\rangle \underset{\text{state-field correspondence}}{=} \sum_a c_a(x)\,\mathcal{O}_a(0)|0\rangle$$

2) The Higgs chiral ring and the Higgs branch

<u>Theorem</u>: Let $R_H = \{\hat{B}_R^{\uparrow} \mid R \in \frac{1}{2}\mathbb{N}_0\}$ be the set of superconformal primaries of $\hat{B}_R$ $(R \in \frac{1}{2}\mathbb{N}_0)$. The OPE of $\cancel{\hat{B}_R^{\uparrow}}$ two elements of R_H is non-singular in the limit of coinciding points and

$$\lim_{x\to 0} \hat{B}_R^{\uparrow}(x)\,\hat{B}_S^{\uparrow}(0) \in R_H \qquad (*)$$

Proof: The OPE yields

$$\hat{B}_R^{\uparrow}(x_1)\,\hat{B}_S^{\uparrow}(x_2)\Big|_{sing} = \sum_{h<0} c_h(x)\,|x_1-x_2|^h\,\mathcal{O}_h(x_2), \quad \text{where } c_h(\lambda x_1, \lambda x_2) = c_h(x_1,x_2) \text{ and}$$

$$\mathcal{O}_h(\lambda x) = \lambda^{-\Delta_h}\,\mathcal{O}_h(x) \qquad (\text{i.e. } \mathcal{O}_h \text{ has conformal weight } \Delta_h)$$

Rescaling by λ yields $\lambda^{-2(R+S)}\,\hat{B}_R^{\uparrow}(x_1)\,\hat{B}_S^{\uparrow}(x_2)\Big|_{sing} = \sum_{h<0} \lambda^h c_h(x)\,|x_1-x_2|^h\,\lambda^{-\Delta_h}\,\mathcal{O}_h(x_2)$

$$\Rightarrow \Delta_h = h + 2(R+S) \;\Rightarrow\; \Delta_h < 2(R+S) \text{ for } h<0$$

$\mathcal{O}_h$ belongs to multiplet labelled by $\Delta', R', r', j', \bar{j}'$ with $R' \geq R+S$

We find $2(R+S) > \Delta_h \geq \Delta' \geq 2R' \geq 2(R+S)$ $\Rightarrow$ no such operator exists

unitarity bounds for any representation

$$\lim_{x \to 0} \hat{B}_R^{\uparrow}(x)\, \hat{B}_S^{\uparrow}(0) = \mathcal{O}_0(0) \quad \text{with } \Delta_0 = 2(R+S) \text{ and } R_0 = R+S, \; j_0 = \bar{j}_0 = r_0 = 0$$

$$\Rightarrow \mathcal{O}_0(0) = \hat{B}^{\uparrow}_{R+S}(0)$$

$\square$

<u>Def:</u> R_H together with the multiplication (*) is a commutative ring called the <u>Higgs chiral ring</u>.

<u>Def:</u> Let R be a commutative ring.

1) The <u>spectrum of R</u> $\mathrm{Spec}(R)$ is

$$\mathrm{Spec}(R) := \{ P \subset R \mid P \text{ is prime ideal} \}$$

2) Let $f \in R$. $D_f := \{ P \in \mathrm{Spec}(R) \mid f \notin P \}$

3) The set $\{ D_f \mid f \in R \}$ is a basis for the <u>Zariski</u> <u>topology</u> on $\mathrm{Spec}(R)$.

<u>Lemma:</u> The Zariski topology is a topology on $\mathrm{Spec}(R)$.

<u>Def:</u> The <u>Higgs branch</u> $\mathcal{M}_H$ is defined by $\mathcal{M}_H = \mathrm{Spec}(R_H)$ (as a scheme).

<u>Conjecture:</u> $\mathcal{M}_H$ is always an affine complex algebraic variety and $R_H = \mathbb{C}[\mathcal{M}_H]$.

<u>Remark:</u> variety $\leftrightarrow$ scheme

Let (V, τ_V) a variety. Define $X = \{ W \subset V \mid W \text{ irred., closed} \}$ and for $U \in \tau_V$ $P_U = \{ W \in X \mid W \cap U \neq \emptyset \}$. Then (X, τ_X) with $\tau_X = \{ P_U \mid U \in \tau_V \}$ is the topological space corresponding to the scheme of V. We find V by restriction to closed points: $V = \{ x \in X \mid \bar{x} = x \}$.

<u>Remark:</u> 1) The conjecture is easy to prove in theories with a Lagrangian description.

2) From physics arguments much richer structures are expected

$\rightsquigarrow$ Hyperkähler quotient

3) Example: $\mathcal{N}=2$ $SU(2)$ gauge theory with 2 hypermultiplets in the doublet

Remark: 1) Physics definition of the Higgs branch:

The Higgs branch of the moduli space of vacua is parametrized by vacuum expectation values of the superconformal primaries of $\hat{B}_R$.

2) In lagrangian theories these operators are built from scalars in the hyper multiplet

3) Moduli space is found by minimizing the potential

$\Rightarrow$ polynomial conditions on fields (**)

In our example we can parametrize the Higgs branch by complex scalars Q^a_I ($a \in \{1,2\}$ (gauge $SU(2)$), $I \in \{1,\dots,2N_f = 4\}$ (flavor $SO(4)$)).

Rewrite $\underset{I}{SO(4)} = \underset{A}{SU(2)} \times \underset{\dot{A}}{SU(2)}$ and use gauge invariant quantities:

$$M_{AB} = Q^a_{A\dot{A}} Q^b_{B\dot{B}} \varepsilon_{ab} \varepsilon^{\dot{A}\dot{B}}$$
$$M_{\dot{A}\dot{B}} = Q^a_{A\dot{A}} Q^b_{B\dot{B}} \varepsilon_{ab} \varepsilon^{AB}$$

(using sum convention)

$$(**) \Rightarrow \begin{cases} M_{AB} M_{CD} \varepsilon^{AC} \varepsilon^{BD} = 0 \\ M_{\dot{A}\dot{B}} M_{\dot{C}\dot{D}} \varepsilon^{\dot{A}\dot{C}} \varepsilon^{\dot{B}\dot{D}} = 0 \\ M_{AB} M_{\dot{A}\dot{B}} = 0 \end{cases}$$

Define $A = M_{11}$, $B = M_{22}$, $C = M_{12} = M_{21}$, $X = M_{\dot{1}\dot{1}}$, $Y = M_{\dot{2}\dot{2}}$, $Z = M_{\dot{1}\dot{2}} = M_{\dot{2}\dot{1}}$

$$\Rightarrow AB = C^2 \wedge XY = Z^2 \wedge \left((A,B,C) = 0 \vee (X,Y,Z) = 0 \right)$$

$$\Rightarrow \mathcal{M}_H = \mathbb{C}^2/\mathbb{Z}_2 \underset{\uparrow}{\wedge} \mathbb{C}^2/\mathbb{Z}_2$$

"disjoint union except for point 0"